SENTENTIAL CALCULUS FOR LOGICAL FALSEHOODS

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Several axiomatic systems for sentential calculus have been developed. Such systems are generally motivated by a consideration of logically true sentences of the formal language. In this paper I present a finitely axiomatized system of sentential calculus for logically false sentences.

1. Introduction. Consider a formal language $L$ with the following symbols:

Sentential variables: $P_1, P_2, \ldots$

Sentential connectives: $\land$—“and,” $\lor$—“or,” $\neg$—“not”

Punctuation: $(),$ and $)$

I will assume the standard definition for “sentence of $L$.” The meta-symbols $R, R_1, R_2, \ldots$ will be used to refer to sentences of $L$. In addition, I will presuppose the standard theory of two-valued truth tables. I will say that a sentence of $L$ is logically true (LT) if and only if the final column of its truth table has only T’s. I will say that a sentence of $L$ is logically false (LF) if and only if the final column of its truth table has only F’s. I will say that two sentences $R_1$ and $R_2$ of $L$ are logically equivalent ($R_1 \equiv R_2$) if and only if the sentence $(R_1 \lor R_2) \land (\neg R_1 \lor \neg R_2)$ is LT.

2. The System SCT. In [1], Hilbert and Ackermann present an axiomatic system of sentential calculus for logical truths. With some small notational differences, their system uses the symbols mentioned above and in addition the symbol “→”. As they note, however, this symbol is to be considered an abbreviation; if $R_1$ and $R_2$ are any two sentences of the language, then $R_1 \rightarrow R_2$ is to be considered an abbreviation for the sentence $\neg R_1 \lor R_2$ ([1], pp. 27-28). In discussing their system, I will eliminate this abbreviation. Since their system is primarily concerned with LT sentences, I will refer to their system as SCT (sentential calculus for truths). With slight notational differences and the removal of the symbol “→”, the Hilbert and Ackermann system may be presented as follows:

Axioms:

(ta) $\neg(P_1 \lor P_1) \lor P_1$

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(tb) \( \neg P_1 \lor (P_1 \lor P_2) \)
(tc) \( (P_1 \lor P_2) \lor (P_2 \lor P_1) \)
(td) \( (\neg (P_1 \lor P_2) \lor (\neg (P_3 \lor P_1) \lor (P_3 \lor P_2)) \)

Rules of Proof:

1. Rule of substitution: We may substitute in a given sentence of \( L \) for a sentential variable any sentence of \( L \), providing that the substitution is made wherever that sentential variable occurs.
2. Rule of \( T \)-implication: From two sentences \( R_1 \) and \( \neg R_1 \lor R_2 \) the sentence \( R_2 \) may be obtained.

I will assume the standard definition of "proof in \( SCT \)." If the sentence \( R \) of \( L \) is provable in \( SCT \), I will write \( \vdash_T R \). Let \( \lambda \) be a set of sentences of \( L \). If \( R \) is provable in \( SCT \) from the axioms augmented by the members of \( \lambda \), I will write \( \lambda \vdash_T R \). Using these notions, the following theorems may be proved about \( SCT \) ([1], chapter 1):

Theorem A: If \( R \) is any sentence of \( L \) and \( \vdash_T R \), then \( R \) is LT.

Theorem B: \( SCT \) is consistent in the sense that there is no sentence \( R \) of \( L \) such that both \( \vdash_T R \) and \( \vdash_T \neg R \).

Theorem C: The axioms ta-td are independent. That is, it is not possible to prove any one of the axioms from all of the others.

Theorem D: The system \( SCT \) is complete in the sense that if \( R \) is any sentence of \( L \) that is LT, then \( \vdash_T R \).

Theorem E: The system \( SCT \) is complete in the sense that if a sentence of \( L \) that is not provable from the axioms is added to the system as an axiom, the new system is inconsistent.

3. The System \( SCF \). I will now present a system of sentential calculus for \( LF \) sentences of \( L \); I will refer to this system as \( SCF \). I will prove theorems about \( SCF \) that are analagous to the theorems presented above about \( SCT \).

Axioms:

(fa) \( \gamma(P_1 \& P_2) \& P_1 \)
(fb) \( \gamma(P_1 \& (P_1 \& P_2)) \)
(fc) \( \gamma(P_1 \& P_2) \& (P_2 \& P_3) \)
(fd) \( \gamma(P_1 \& P_2) \& (\gamma(P_3 \& P_1) \& (P_3 \& P_2)) \)

Rules of Proof:

1. Rule of substitution: same as that for \( SCT \).
2. Rule of \( F \)-implication: From the two sentences \( R_1 \) and \( \gamma R_1 \& R_2 \) the sentence \( R_2 \) may be obtained.

I will again assume the standard definition of "proof in \( SCF \)." If the sentence \( R \) of \( L \) is provable in \( SCF \), I will write \( \vdash_F R \). As for \( SCT \), if \( \lambda \) is a set of sentences of \( L \), I will write \( \lambda \vdash_F R \) when \( R \) is provable in \( SCF \) from the axioms augmented by the members of \( \lambda \). Theorems analogous to those presented above for \( SCT \) could be proven in a straightforward manner for
**SCF** without reference to **SCT**. However, I will construct proofs for such theorems (except Theorem A') by relating them to the theorems for **SCT**.

In the following material, I will make use of the one-to-one function $F$ from the set of sentences of $L$ onto the set of sentences of $L$, defined in the following way:

(a) $F(P_i) = P_i$, for any sentential variable $P_i$
(b) $F(\gamma R) = \gamma F(R)$, for any sentence $R$ of $L$
(c) $F(R_1 \lor R_2) = F(R_1) \lor F(R_2)$, for any sentences $R_1$ and $R_2$ of $L$
(d) $F(R_1 \land R_2) = F(R_1) \land F(R_2)$, for any sentences $R_1$ and $R_2$ of $L$

Theorem 1: Let $R$ be a sentence of $L$. Then $F(F(R)) = R$.

Proof: By induction on the number, $n$, of connectives in $R$; $n$ will be called the length of $R$. Suppose $n = 0$. Then $R$ is a sentential variable, say $P_i$. Then by (a), $F(F(P_i)) = F(P_i) = P_i$. Suppose the theorem is true for all $n$ less than some number $p$, $p$ greater than 0. We must show that the theorem holds for $p$. Let $R$ be an arbitrary sentence of length $p$. Then $R$ must be of the form $\gamma R_1$, $R_1 \lor R_2$, or $R_1 \land R_2$. Suppose for some $R_1$, $R = \gamma R_1$. Then by (b), $F(F(R)) = F(F(\gamma R_1)) = \gamma F(F(R_1))$. But $R_1$ has length $p - 1$, and thus by assumption the theorem holds for $R_1$. Thus $F(F(R_1)) = R_1$. Thus $F(F(R)) = \gamma R_1 = R$. Similarly it is easy to show that if $R$ has the form $R_1 \lor R_2$ or $R_1 \land R_2$ then the theorem holds. Thus the theorem holds for $n = p$, and hence for all $n$. The following lemmas will be needed for the proof of the next theorem.

Lemma 1: If $R_1$ is the result of substituting $R_2$ in $R_3$ for the sentential variable $P_i$, then $F(R_1)$ is the result of substituting $F(R_2)$ in $F(R_3)$ for $P_i$.

Proof: By induction on the length, $n$, of $R_3$. Suppose $n = 0$. Then $R_3$ must be a sentential variable. Suppose $R_3$ is $P_i$. Then $R_1 = R_2$, and $F(R_1) = F(R_2)$. But $F(R_3) = P_i$. Thus the result of substituting $F(R_2)$ for $P_i$ in $F(R_3)$ is just $F(R_2)$. Hence $F(R_1)$ is the result of substituting $F(R_2)$ in $F(R_3)$ for $P_i$. Suppose $R_3$ is $P_j$, where $P_i \neq P_j$. Then $R_1 = R_3$, and thus $R_1$ is $P_i$. But then both $F(R_1)$ and $F(R_3)$ are just $P_i$. Hence, $F(R_1)$ is the result of substituting $F(R_2)$ in $F(R_3)$ for $P_i$. Thus the lemma holds for $n = 0$. Now, suppose the lemma is true for all $n$ less than some number $p$, where $p$ is greater than 0. Let $R_3$ be an arbitrary sentence of $L$ of length $p$.

Case 1: $R_3 = \gamma R_4$, for some $R_4$. Then $R_1 = \gamma R_5$, for some $R_5$, where $R_5$ is the result of substituting $R_2$ in $R_4$ for the sentential variable $P_i$. But $R_4$ is of length $p - 1$. Hence by induction hypothesis, $F(R_3)$ is the result of substituting $F(R_2)$ in $F(R_4)$ for the sentential variable $P_i$. But $F(R_3)$ is just $F(\gamma R_4)$ which is $\gamma F(R_4)$; further, $F(R_3)$ is just $F(\gamma R_5)$ which is $\gamma F(R_5)$. Thus $F(R_1)$ is the result of substituting $F(R_2)$ in $F(R_3)$ for $P_i$.

Case 2: $R_3 = R_4 \lor R_7$, for some $R_4$ and $R_7$. Then $R_1 = R_6 \lor R_7$, where $R_6$ is the result of substituting $R_2$ in $R_4$ for $P_i$ and $R_7$ is the result of substituting $R_2$ in $R_7$ for $P_i$. Since the length of $R_4$ is less than $p$ and the length of $R_7$ is less than $p$, we have $F(R_6)$ is the result of substituting $F(R_2)$ in $F(R_4)$ for $P_i$ and $F(R_7)$ is the result of substituting $F(R_2)$ in $F(R_7)$ for $P_i$. 

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**SENTENTIAL CALCULUS** 349
But \( F(R_1) \) is just \( F(R_a \lor R_7) \) which is \( F(R_a) \land F(R_7) \); further, \( F(R_3) \) is just \( F(R_a \lor R_b) \) which is \( F(R_4) \land F(R_5) \). Thus \( F(R_1) \) is the result of substituting \( F(R_a) \) in \( F(R_3) \) for \( P, \) .

**Case 3:** \( R_3 = R_4 \lor R_5 \), for some \( R_4 \) and \( R_5 \). This case is exactly the same as Case 2, interchanging "\&" and "\lor".

Thus the lemma is true for \( n = p \), and hence it is true for all \( n \).

**Lemma 2:** If \( F(R_1) \) is the result of substituting \( F(R_a) \) in \( F(R_3) \) for the sentential variable \( P, \) then \( R_1 \) is the result of substituting \( R_a \) in \( R_3 \) for \( P, \) .

**Proof:** Suppose the hypothesis of the lemma is true. Then by Lemma 1, \( F(F(R_1)) \) is the result of substituting \( F(F(R_a)) \) in \( F(F(R_3)) \) for \( P, \) . But then by Theorem 1, \( R_1 \) is the result of substituting \( R_a \) in \( R_3 \) for \( P, \) .

The next theorem will be of fundamental importance in the work to follow. It simply tells us that if we have a proof in either \( \text{SCF} \) or \( \text{SCT} \), then we can transform it into a proof in the other system by means of the function \( F, \) .

**Theorem of Proof Correspondence:** Let \( R_1, R_2, \ldots, R_n \) be a series of sentences of \( L, \) and let \( F(R_1), F(R_2), \ldots, F(R_n) \) be a series of sentences of \( L \) obtained from the first by taking the \( F \)-transformation of each sentence in that series. Then \( R_1, R_2, \ldots, R_n \) constitutes a proof of \( R_n \) in \( \text{SCT} \) if and only if \( F(R_1), F(R_2), \ldots, F(R_n) \) constitutes a proof of \( F(R_n) \) in \( \text{SCF} \).

**Proof:** By induction on the number, \( n, \) of steps in the proof; \( n \) will be called the length of the proof. The theorem may be broken into two parts.

**First** suppose \( R_1, R_2, \ldots, R_n \) is a proof in \( \text{SCT} \) of \( R_n \). We want to show that \( F(R_1), F(R_2), \ldots, F(R_n) \) is a proof of \( F(R_n) \) in \( \text{SCF} \). Suppose \( n = 1 \). Then \( R_1 \) is an axiom of \( \text{SCT} \). But \( fa = F(ta), \ldots, \) and \( fd = F(td) \). Hence \( F(R_1) \) is a proof of \( F(R_1) \) in \( \text{SCF} \), and thus this half of the theorem holds for \( n = 1 \). Now, suppose this first half of the theorem holds for all \( n \) less than some number \( p, \) where \( p \) is greater than 1. We must show that this half of the theorem holds for \( p \). Suppose \( R_1, R_2, \ldots, R_p \) is a proof in \( \text{SCT} \) of \( R_p \). We want to show that \( F(R_1), F(R_2), \ldots, F(R_p) \) is a proof in \( \text{SCF} \) of \( F(R_p) \). The only question that may arise concerns step \( p, \) for by the induction hypothesis, the steps through \( p - 1 \) constitute a proof. Consider the justification for step \( p, \) .

**Case 1:** \( R_p \) is an axiom. This case is the same as for \( n = 1 \).

**Case 2:** \( R_p \) follows by substitution of the sentence \( R \) in \( R_i \) for the sentential variable \( P, \) . Then by Lemma 1, \( F(R_p) \) follows by substitution of the sentence \( F(R) \) in \( F(R_i) \) for the sentential variable \( P, \) .

**Case 3:** \( R_p \) follows by \( T \)-implication from two previous sentences \( R_i \) and \( R_j, \) where \( R_j = R_i \lor R_p. \) But then \( F(R_i) \) is just \( F(R_i) \lor F(R_p) \) which is \( F(R_i) \lor F(R_p). \) Hence \( F(R_p) \) follows from \( F(R_i) \) and \( F(R_j) \) by \( F \)-implication.

Thus the first half of the theorem holds for \( n = p, \) and hence for all \( n. \) The proof of the second half of the theorem relies on the fact that \( F(F(R)) = R \) (Theorem 1) and is exactly similar to the above proof, using Lemma 2 in place of Lemma 1.
Corollary 1: \(\vdash \tau R \text{ if and only if } \vdash \phi F(R)\).

Corollary 2: Let \(\lambda\) be a set of sentences, and let \(F(\lambda)\) be the set of sentences whose members are the \(F\)-transforms of the members of \(\lambda\). Then \(\lambda \vdash \tau R \text{ if and only if } F(\lambda) \vdash \tau F(R)\).

\textbf{Proof:} Note that a step \(R_i\) in the \(\text{SCT}\) proof is a member of \(\lambda\) if and only if \(F(R_i)\) is a member of \(F(\lambda)\). The proof is then the same as the proof for the Theorem of Proof Correspondence.

I will now proceed to prove theorems analogous to the first three presented for \(\text{SCT}\), above.

\textbf{Theorem A'}: If \(R\) is any sentence of \(L\) and \(\vdash \tau R\), then \(R\) is \(\text{LF}\).

\textbf{Proof:} The proof is exactly parallel to the proof for Theorem A. Note that all of the axioms for \(\text{SCF}\) are \(\text{LF}\), and that the rules of proof preserve the property of being \(\text{LF}\). The theorem then follows immediately.

\textbf{Theorem B'}: \(\text{SCF}\) is consistent in the sense that there is no sentence \(R\) of \(L\) such that \(\vdash \tau R\) and \(\vdash \tau \neg R\).

\textbf{Proof:} Suppose there were a sentence \(R\) such that \(\vdash \tau R\) and \(\vdash \tau \neg R\). Then by Theorem 1, \(\vdash \tau F(F(R))\) and \(\vdash \tau \neg F(F(R))\). Then by definition of \(F\), \(\vdash \tau F(\neg F(R))\). Then by Corollary 1, \(\vdash F(R)\) and \(\vdash \tau F(R)\). But this contradicts Theorem B. Hence there is no sentence \(R\) of \(L\) such that \(\vdash \tau R\) and \(\vdash \tau \neg R\).

\textbf{Theorem C'}: The axioms \(fa-fd\) are independent. That is, it is not possible to prove any one of the axioms from all of the others.

\textbf{Proof:} Suppose the theorem were false, and that there is a proof of one of the axioms from the others. Note that \(ta = F(fa), \ldots, \text{ and } td = F(fd)\). Then by the Theorem of Proof Correspondence, one could transform the \(\text{SCF}\) proof into an \(\text{SCT}\) proof in which one of the \(\text{SCT}\) axioms is proven from the others. But this contradicts Theorem C. Hence the axioms \(fa-fd\) are independent.

I will prove a few intermediate theorems before proving the remaining analogous theorems.

\textbf{Theorem 2}: For any sentence \(R\) of \(L\), if \(R\) is \(\text{LT}\), then \(F(R)\) is \(\text{LF}\).

\textbf{Proof:} Suppose \(R\) is \(\text{LT}\). Then by Theorem D, \(\vdash \tau R\). Then by Corollary 1, \(\vdash \tau F(R)\). Then by Theorem A', \(F(R)\) is \(\text{LF}\).

\textbf{Theorem 3}: If \(F(R)\) is \(\text{LF}\), then \(R\) is \(\text{LT}\), for \(R\) an arbitrary sentence of \(L\).

\textbf{Proof:} Suppose \(F(R)\) is \(\text{LF}\). Then \(\tau F(R)\) is \(\text{LT}\). Thus \(\tau F(\neg R)\) is \(\text{LT}\). Then by Theorem 2, \(F(F(\tau R))\) is \(\text{LF}\). By Theorem 1, \(\tau R\) is \(\text{LF}\). Hence \(R\) is \(\text{LT}\).

\textbf{Theorem 4}: Let \(R, R_1,\) and \(R_2\) be sentences of \(L\). The following are equivalent:

(a) \(R\) is \(\text{LT}\) if and only if \(F(R)\) is \(\text{LF}\),
(b) \(R\) is \(\text{LF}\) if and only if \(F(R)\) is \(\text{LT}\),
(c) \(R_1 \text{ LE } R_2\) if and only if \(F(R_1) \text{ LE } F(R_2)\).
Proof: (a) implies (b): Suppose (a) is true. Let \( R \) be an arbitrary sentence of \( L \) that is LF. Then \( \forall R \) is LT. By (a), \( F(\forall R) \) is then LF, and hence \( \forall F(R) \) is LF. Thus \( F(R) \) is LT. Let \( R \) be an arbitrary sentence of \( L \) such that \( F(R) \) is LT. Then by (a), \( F(F(R)) \) is LF. By Theorem 1, \( R \) is LF.

(b) implies (c): Suppose (b) is true. Let \( R_1 \) and \( R_2 \) be arbitrary sentences of \( L \) such that \( R_1 \leq F(R_2) \). Then \( (R_1 \land R_2) \lor (\forall R_1 \land \forall R_2) \) is LT. Thus \( F((R_1 \land R_2) \lor (\forall R_1 \land \forall R_2)) \) is LT, by Theorem 1. By (b), \( F((R_1 \land R_2) \lor (\forall R_1 \land \forall R_2)) \) is LF, and thus \( \forall F((R_1 \land R_2) \lor (\forall R_1 \land \forall R_2)) \) is LT. But then \( (F(R_1) \land F(R_2)) \lor (\forall F(R_1) \land \forall F(R_2)) \) is LT. Hence, \( F(R_1) \leq F(F(R_2)) \). To prove \( R_1 \leq R_2 \) if \( F(R_1) \leq F(R_2) \), it is only necessary to reverse the steps in the above argument.

(c) implies (a): Suppose (c) is true. Assume \( R \) is LT. Then \( \forall R \) is \( \forall P_1 \lor \forall P_1 \). Thus by (c), \( F(R) \leq F(P_1 \lor \forall P_1) \). But \( F(P_1 \lor \forall P_1) \) is just \( P_1 \lor \forall P_1 \), which is LF. Thus \( F(R) \) is LF. Now, assume \( F(R) \) is LF. Then \( F(R) \leq P_1 \lor \forall P_1 \). By (c), \( F(F(R)) \leq F(P_1 \lor \forall P_1) \). But \( F(F(R)) = R \) by Theorem 1, and \( F(P_1 \lor \forall P_1) = P_1 \lor \forall P_1 \). Thus \( R \leq P_1 \lor \forall P_1 \). Hence \( R \) is LT.

Theorem 5: \( R \) is LF if and only if \( F(R) \) is LT, for any sentence \( R \) of \( L \).

Proof: The theorem follows directly from Theorems 2, 3, and 4.

Theorem 6: \( R_1 \leq R_2 \) if and only if \( F(R_1) \leq F(R_2) \), for any sentences \( R_1 \) and \( R_2 \) of \( L \).

Proof: The theorem follows directly from Theorems 2, 3, and 4.

I will now prove the remaining two analogue theorems.

Theorem D': SCF is complete in the sense that if a sentence \( R \) of \( L \) is LF, then \( \forall R \).

Proof: Suppose \( R \) is a sentence of \( L \) that is LF. Then by Theorem 5, \( F(R) \) is LT. Thus by Theorem D, \( \forall F(R) \). By Corollary 1, \( \forall F(F(R)) \). By Theorem 1, \( \forall F(R) \).

Theorem E': SCF is complete in the sense that if a sentence of \( L \) that is not provable from the axioms is added to the system as an axiom, the new system will be inconsistent.

Proof: Let \( R \) be a sentence of \( L \) such that not \( \forall R \). Then \( R \) is not LF, by Theorem D'. By Theorem 5, \( F(R) \) is not LT. Add \( F(R) \) to SCT as an axiom. By Theorem E, the new system is inconsistent. That is, for some sentence \( R_1 \) of \( L \), \( \{F(R)\} \vdash R_1 \) and \( \{F(R)\} \vdash \forall R_1 \). By Corollary 2, \( \{F(F(R))\} \vdash F(R_1) \) and \( \{F(F(R))\} \vdash F(\forall R_1) \). But since \( F(\forall R_1) = \forall F(R_1) \), \( \{F(F(R))\} \vdash \forall F(R_1) \). By Theorem 1, \( \{R\} \vdash F(R_1) \) and \( \{R\} \vdash \forall F(R_1) \). Thus the new system is inconsistent.

It seems then that the system SCF has all of the "nice" logical properties that the system SCT possesses.

4. Further Comments. The obvious next step to take is to formulate a system of predicate calculus for logically false sentences. This work is currently in progress.
Another interesting problem that arises in connection with this study is developing an analogous finite axiomatization for logically contingent (LC) sentences. There are many difficulties. For example, consider possible rules of proof for such a system. Clearly the rule of substitution cannot be used as it stands. \( P_1 \lor P_2 \) is LC, but the result of substituting \( \forall P_1 \) for \( P_2 \) in that sentence would not be LC. This difficulty suggests a rule of substitution something like the following: We may substitute in a sentence of \( L \) for a sentential variable any LC sentence of \( L \) that has no sentential variables in common with the original sentence except perhaps the variable for which substitution is being made.

Other difficulties with rules of proof arise when one considers what possible deductions could be made from \( R_1 \land R_2 \) or \( R_1 \lor R_2 \). If all we know about \( R_1 \) is that it is LC (has occurred in the proof), then we can conclude nothing about \( R_2 \) from either the fact that \( R_1 \land R_2 \) is LC (has occurred in the proof) or the fact that \( R_1 \lor R_2 \) is LC (has occurred in the proof). This does suggest a rule something like: From \( R_1 \land R_2 \) and \( R_1 \lor R_2 \), both \( R_1 \) and \( R_2 \) may be obtained.

Some oddities also arise in the meta-characterization of such a system. We would not want SCC to be consistent in the sense that there is no sentence such that both it and its negation are provable in the system. On the contrary, if a sentence is provable, then we want its negation to be provable as well, for if \( R \) is LC, then so is \( \neg R \). Thus if the system is complete in the sense that for any LC sentence \( R \) of \( L \), \( R \) is provable in SCC, then the system must be inconsistent in the above sense. However, we would want SCC to be consistent in the sense that not every sentence of \( L \) is provable in the system.

There are several points that lead me to suspect that SCC is finitely axiomatizable, in spite of these and other difficulties. First, of course, is the fact that SCT and SCF are both finitely axiomatizable. Secondly, the set of LC sentences of \( L \) is completely decidable by truth tables. It would seem odd to have a completely decidable set of sentences that was not finitely axiomatizable.

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