A classification of logical systems


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ON ARISTOTELIAN SYLLOGISTIC

This article is dedicated to Professor Jan Łukasiewicz

§ 1. Let us recapitulate at first the results of investigations of J. Łukasiewicz concerning Aristotelian syllogistic which are referred to in this article.

To denote general- and particular-affirmative expressions Łukasiewicz introduced the respective symbols \( \text{Uab} \) and \( \text{lab} \). Negative expressions — general and particular — he defined as \( \text{Nlab} \) and \( \text{NUab} \), respectively.

1 These results are presented in the following publications of Jan Łukasiewicz:

Znaczenie analizy logicznej dla poznania (The importance of logical analysis for cognition), Przegląd Filozoficzny, Warszawa 1931, p. 379.

Elementy logiki matematycznej (Elements of mathematical logic) Authorised by Łukasiewicz typographed copy of his lectures, edited by M. Presburger, Warszawa 1939.


In the last article we find among others a historical approach to the problems considered here.

It is my duty to point out that in the course of my researches, the results of which I am presenting here, I made use not only of the above publications of professor Łukasiewicz but also of his many valuable suggestions and remarks, for which I owe him my sincerest gratitude.

2 I will use in my article Łukasiewicz's notation. In this notation the letters \( C, K \) and \( N \) are used as signs of implication, conjunction and negation, respectively. Functors are always preceding arguments. In the expression \( CK \text{Uab} \text{Use} \text{Uab} \) for instance the first letter \( C \) indicates that this expression is an implication and the letter \( K \) — that the conjunction of \( \text{Uab} \) \text{Use} \) is the antecedent of this implication; its consequent is the expression \( \text{Uab} \).
Further, Łukasiewicz has shown that the laws of the square of opposition, those of conversion and all true Aristotelian syllogisms can be deduced from the following four expression:

(1) \( Uaa \)  
(2) \( Iaa \)  
(3) \( CKUcbUacUab \)  
(4) \( CKUcbIcalab \)

The expressions (1) — (4) are therefore axioms of these systems.

§ 2. We shall formulate now the problems dealt with in this article in such a way as to explain their intuitive meaning. For this purpose the following definitions are required.

Definition I. By a meaningful expression of AL,\(^3\) we shall mean any expression obtainable from any meaningful expression of the calculus of propositions by way of substituting for all its variables general- or particular-affirmative expressions. Thus the sole functors occurring in meaningful expressions are those of the calculus of propositions and the functors \( U \) and \( I \). The remaining functors of Aristotelian syllogistic may be defined — as mentioned above — in terms of \( U \), \( I \) and the sign of negation.

Definition II. Let \( k \) be a natural number. A meaningful expression of AL will be called a \( k \)-range consequence of a given finite sequence of meaningful expressions of AL

\[
x_1, x_2, \ldots, x_n
\]

in the case \( k=1 \), if it is obtainable from the expressions (5) and from the theses of the calculus of propositions by applying the rule of substitution and by applying once the rule of implication;

in the case \( k>1 \), if it is a \( 1 \)-range consequence of such a sequence of meaningful expressions of AL which contains as its elements solely \( 1 \)-range consequences of the expressions (5), where \( 1\leq k \).

\(^3\) The letters AL are an abbreviation of the expression: Aristotelia's Logica.

Any expression which, is a \( k \)-range consequence of the expression (5) for a given natural \( k \), will be called shortly a consequence of the expressions (5).

Definition III. A meaningful expression of AL which is a consequence of the expressions (1) — (4) will be called a thesis of AL.

Definition IV. A meaningful expression of AL \( x \) will be called a rejected expression with respect to a given finite sequence (5) of meaningful expressions of AL if for some natural \( i \) fulfilling the condition \( 1\leq i \leq n \) the expression \( x_i \) is a consequence of the expressions (1) — (4) and of \( x \).

As we shall have to speak often of expressions rejected with respect to the expression

\[
CKUbcUacUlab
\]

let us introduce the following terminological abbreviation.

Definition V. A meaningful expression of AL will be called a rejected expression if it is a rejected expression with respect to the expression (6).

Let us consider now the intuitive meaning of the definitions IV and V.

First of all let us note that if each of the expressions (5) referred to in Definition IV is a false expression all rejected expressions with respect to (5) will also be false.

Let us also note that the expression (6) referred to in Definition V and which has the form of an Aristotelian syllogism seems intuitively false and that it is treated as a false expression in traditional logic.\(^4\)

Hence all rejected expressions seem also false.

Following Łukasiewicz’s terminology we shall call any sequence of meaningful expressions of AL a sequence of rejected axioms if every false expression of AL is a rejected expression with respect to this sequence.

Before formulating in detail the problems discussed in this paper let us note that we shall not confine our considerations to Aristotelian syllogisms, whose structure is defined by way of

\(^4\) We may add here that expression (6) is one of two expressions regarded by Łukasiewicz as „axiomatically rejected”.
certain more or less artificial conditions, but that they will deal with all meaningful expressions of AL. Some meaningful expressions of AL, such as for instance the expressions

\[ CKUaKbKcKdUaKbKcKd \]

and \[ CKIdKcKdKb \]
do not have the form of Aristotelian syllogisms though they seem to be as interesting as these syllogisms. In connection with this arises the problem of generalising Lukasiewicz’s results concerning Aristotelian syllogisms.

Our first task will be to investigate whether the set of expressions (1) – (4) from which, as we know, all true Aristotelian syllogisms may be deduced is also a sufficient basis for the deduction of all true meaningful expressions of AL.

A negative answer to this question would induce us either to search for another finite set of true meaningful expressions of AL having the above property or to prove that no such set exists.

Several difficulties arise in connexion with this problem.

First of all, the number of meaningful expressions of AL is infinite whereas the number of Aristotelian syllogisms is finite. There is also no criterion at our disposal by means of which we could always distinguish true meaningful expressions of AL from the false ones. It is well known that such criteria often permit the solution of problems similar to our problem, when investigating the methodology of the calculus of propositions.

Secondly, since the system of all meaningful expressions of AL is not a complete system we can not solve our problem by showing that a set of all consequences of any set consisting of expressions (1) – (4) and of any meaningful expression of AL which is not a thesis of AL must contain two contradictory expressions.

But our problem would be solved, and the manner of solving it would be somehow similar to the last mentioned, if we could generalise Lukasiewicz’s thesis that every Aristotelian syllogism which is not a thesis of AL is a rejected and hence a false expression.

Let us note that the notion of rejected expression as used by Lukasiewicz has the same intuitive meaning as the notion introduced by definitions IV and V, notwithstanding differences of the formulations.

Let us formulate now our second fundamental problem: does there exist a finite set of meaningful expressions of AL such that a) we would be inclined for those or some other reasons to consider every expression of this set as false, and that b) every meaningful expression of AL which is neither a consequence of (1) – (4) nor any other finite set of true meaningful expressions of AL would be a rejected expression with respect to this set.

Both our problems may be shortly expressed as follows: does there exist:

1) a set of axioms of the system of all true meaningful expressions of AL and

2) a set of rejected axioms

and if such sets do exist what is their form?

In particular, the question arises whether the set of expressions (1)–(4) is a set of axioms of the system under consideration and whether the set of rejected axioms may be reduced to the sole expression (6) referred to in definition V?

All these problems have been formulated by Lukasiewicz.

The following theorem throws some light on the meaning of the notion of rejected expression.

Theorem 1. No meaningful expression of AL is both a thesis of AL and a rejected expression.

Proof. Should a given meaningful expression of AL be a thesis of AL and, at the same time, a rejected expression then according to definitions II, III, IV, and V the expression (6) referred to in definition V would also be a thesis of AL. Hence our theorem will be proved if we can show that the expression (6) is not a consequence of the expressions (1) – (4). For this purpose we shall present an interpretation turning the expressions (1) – (4) and all their consequences into true propositions and the expression (6) into a false one.

This interpretation is as follows:

For the variables occurring in meaningful expressions of AL we are allowed to substitute solely natural numbers greater than 1. The expressions \( U_{ab} \) and \( I_{ab} \) have to be understood accordingly as

- a is a divisor of b
- a and b have a common divisor greater than 1.
§ 3. Definition VI. A meaningful expression of $\text{AL}$ and a finite sequence of meaningful expressions of $\text{AL}$ (5) are inferentially equivalent if and only if $x$ is a consequence of the sequence of expressions (5) and every expression of this sequence is a consequence of $x$.

Definition VII. The term elementary expression will be used to denote all general- and particular-affirmative expressions and the negations of these expressions.

Definition VIII. Let $k$ be a natural number. The term $k$-range simple conjunction will be used to denote:

- in the case $k = 1$ every elementary expression,
- in the case $k > 1$ every conjunction of an elementary expression and a $(k - 1)$-range simple conjunction.

Every meaningful expression of $\text{AL}$ which for some natural $k$ is a $k$-range simple conjunction will be called shortly a simple conjunction.

Definition IX. Let $k$ be a natural number and $x$ a $k$-range simple conjunction. The term $k$-range argument of $x$ will be used to denote:

- in the case $k = 1$ the expression $x$,
- in the case $k > 1$ the first member of the conjunction $x$ and every argument of the $(k - 1)$-range simple conjunction which is the second member of $x$.

It is easy to see that each argument of any simple conjunction is an elementary expression.

Definition X. A conditional sentence having a simple conjunction as antecedent and an elementary expression as consequent will be called a simple expression.

From the definitions II, IV, V and from the fact that the relation of being a consequence of is transitive we may deduce the following:

Lemma I. If $x$ is a meaningful expression of $\text{AL}$ and $y$ a rejected expression and, at the same time, a consequence of the expression $x$ then the expression $x$ is also a rejected expression.

Lemmas II—IX may be inferred from the above definitions and from the properties of meaningful expressions of the calculus of propositions.

Lemma II. Every meaningful expression of $\text{AL}$ is inferentially equivalent to a finite sequence of meaningful expressions of $\text{AL}$ every member of which is either an elementary or a simple expression.

Lemma III. Each conditional sentence $x$ having a simple conjunction as antecedent and a simple expression $y$ as consequent is inferentially equivalent to a simple expression of which the consequent has the same form as the consequent of $y$ and the antecedent is a simple conjunction having as arguments those, and only those, elementary expressions which have the same form as the arguments of the antecedents of $x$ and of $y$.

Lemma IV. In the following cases the conditional sentence $C^x y^5$ is a thesis of $\text{AL}$:

(IVi) if $x$ is a meaningful expression of $\text{AL}$ and $y$ is a thesis of $\text{AL}$;

(IVii) if $x$ is a simple conjunction some argument of which is a negation of a thesis of $\text{AL}$ and $y$ is a meaningful expression of $\text{AL}$;

(IViii) if $x$ is a simple conjunction, $y$ a meaningful expression of $\text{AL}$ and among the theses of $\text{AL}$ there is a conditional sentence the antecedent of which is equiform to some argument of $x$ and the consequent has the form of $y$.

Lemma V. If the consequents of the simple expressions $x$ and $y$ are equiform and if every argument of the antecedent of $x$ which is not a thesis of $\text{AL}$ has the form of a corresponding argument of the antecedent of $y$ then $y$ is a consequence of $x$.

Lemma VI. Let $x, y, z$ be simple expressions fulfilling the following conditions: 1) one of the arguments of the antecedent of $x$ and the consequent of $y$ are equiform; 2) the consequent of $z$ and the consequent of $x$ are equiform; 3) the antecedent of $y$.

$^5$ The expression $C^x y^5$ is to be understood as the name of the conditional sentence of which $x$ is the antecedent, $y$ — the consequent. An analogous notation will be adopted for the remaining terms of $\text{AL}$. 
z has as arguments those and only those elementary expressions which have the form of the arguments of the antecedent of y and of those arguments of the antecedent of x which do not have the form of the consequent of y; then z is a consequence of x and y.

Lemma VII. If x is a meaningful expression of AL the conditional sentence C*xy will be:

(VIIa) a consequence of x if y is a meaningful expression of AL;

(VIIa) inferentially equivalent to x if y is a thesis of AL;

Lemma VIII. If x, y, z are meaningful expressions of AL and the conditional sentence C*xz is a thesis of AL then:

(VIIIa) the conditional sentence C*xy is a consequence of the conditional sentence C*yz and the expressions (1) — (4);

(VIIIa) the conditional sentence C*zxy is a consequence of the conditional sentence C*zx and the expressions (1) — (4).

Lemma IX. Let α and β be general- or particular-affirmative expressions, x a simple conjunction one argument of which is equivalent to α, and x' a simple conjunction having as arguments: the expression N*β and those and only those elementary expressions which have the form of those arguments of x which do not have the form of α. The conditional sentences C*xβ and C*x'N*α are then inferentially equivalent.

§ 4. Lemma X. The following meaningful expressions of AL are theses of AL.

(Xi) Uaa (Xa) Iaa (Xa) Clablba

(Xa) Ckublab (Xa) Ckublab (Xa) Ckublab Uab

(Xi) CkibUcalab (Xa) CkibUcalab (X) CkibUcalab Uab

(Xi) CkibUcalab (Xa) CkibUcalab (X) CkibUcalab Uab

(Xi) CkibUcalab (Xa) CkibUcalab (X) CkibUcalab Uab

(Xi) CkibUcalab (Xa) CkibUcalab (X) CkibUcalab Uab

Proof. The expressions (Xi) — (Xii) are theses of AL according to § 1. The expressions (Xiii) — (Xia) are theses of AL since the expressions Cpp and Ckpqq Ckstq CksKtp are theses of the calculus of propositions and the expressions (Xii), (Xiii) and (Xia) are theses of AL.

§ 5. Lemma XI. The simple expression

(7) Ckublab

is a rejected expression.

Proof. It follows from Lemma VI that the expression

(8) Ckublab Uab

is a consequence of the expressions (7) and (Xa). From lemma VII it follows that the expression (6) referred to in definition V is a consequence of the expressions (8), (Xa) and (1) — (4). From this and from lemma I follows immediately lemma XI.

Lemma XII. The elementary expressions:

(Xii) NUaa (Xii) Nlaa

(Xii) NUaUab (Xii) Nlaab

are rejected expressions.

Proof. By lemmas I, IVa, Xa and Xb.

Lemma XIII. The following meaningful expressions of AL

(Xiii) Uab (Xiii) Ckublab

(Xiii) Ckublab (Xiii) Ckublab Uab

(Xiii) Ckublab (Xiii) Ckublab Uab

(Xiii) Ckublab (Xiii) Ckublab Uab

are rejected expressions.

Proof. This follows from lemmas I, III, VII, XI and from the fact that the expression (6) referred to in definition V is a rejected expression.

§ 6. Definition XI. Let k be a natural number. By a k-range chain connecting the variables a and b we shall mean:

in the case k = 1 the expression Uab,

in the case k > 1 the conjunction of Uab and a (k — 1)-range chain connecting the variables c and b.

Thus for instance the expressions

KUacUcb and KUacKUcdUab

are respectively 2nd-range and 3rd-range chains connecting the variables a and b.

Definition XII. Of a simple conjunction x we say that it connects the variables a and b if every argument of some
chain (of any range) connecting the variables $a$ and $b$ is equiform to a corresponding argument of $x$.

**Lemma XIV.** If $x$ is a simple expression of which $Uab$ is the consequent, and a simple conjunction connecting the variables $a$ and $b$ the antecedent, then $x$ is a thesis of $AL$.

**Proof.** Let $x$ be an expression fulfilling the conditions of this lemma. In view of lemma V we may assume that the antecedent of $x$ is a $k$-range chain connecting the variables $a$ and $b$. In the case of $k = 1$ lemma XIV follows from $X_{1a}$. Let us suppose that lemma XIV is true for some natural number $j$ and let the antecedent of $x$ be a $(j + 1)$-range chain connecting $a$ with $b$. We may assume (this will not affect the scope of our thesis) $c$ to be the variable occurring in the predicate of the first argument of the antecedent of $x$. The expression $x$ may then be denoted by

$$C^sK^8U^8acyU^8ab$$

$y$ being a $j$-range chain connecting $c$ with $b$. But then the expression

$$C^sU^8cb$$

is a thesis of $AL$. From this and from lemmas VI and $X_4$ it follows that the expression (9) is a thesis of $AL$, q. e. d.

**Lemma XV.** If the sign of negation is not contained in the antecedent of the simple expression $x$ and if $Uab$ is the consequent of $x$, then $x$ is either a thesis of $AL$ or a rejected expression.

**Proof.** Two cases may be distinguished:

(i) the antecedent of $x$ does connect the variables $a$ and $b$;

(ii) the antecedent of $x$ does not connect these variables;

In the case (i) lemma XV follows from lemma XIV.

Passing to case (ii) let us substitute in $x$ the variable $b$ for every variable connected with $b$ by the antecedent of $x$; for the remaining variables occurring in $x$ and differing in form from $b$ let us substitute the variable $a$.

Let $x'$ be the expression obtained thus from $x$. The consequences of the expressions $x$ and $x'$ are obviously equiform.

Next let us note that none of the arguments of the antecedent of $x$ is identical with the expression (XIII$_1$).

This follows from our assumption that in case (ii) the antecedent of $x$ does not connect the variables $a$ and $b$.

We shall prove now that neither is the expression (XIII$_b$) obtainable from some general-affirmative argument $a$ of the antecedent of $x$ by applying substitution. For this purpose let us consider first the case where the variable occurring in the predicate of $a$ is identical with $b$ or is connected with $b$ by the antecedent of $x$. In that case an expression which is a result of substitution applied to $a$ must also contain in its subject the variable $b$. If however the variable occurring in the predicate of $a$ is neither identical with $b$ nor connected with $b$ by the antecedent of $x$, an expression which is a result of substitution upon $a$ must contain in the predicate the variable $a$.

Thus we have proved that in none of the possible cases does the expression (XIII$_1$) occur in the antecedent of $x$.

In the antecedent of $x'$ may however occur the following expressions:

$$Uaa, Ubb, Iaa, Ibb, Uba, lab and Iba.$$

From this and from the lemmas V, VI and $X_{1a,b}$ it follows that either the expression (7) or the expression (XIII$_b$) is a consequence of $x'$ and hence of $x$.

From this and from the lemmas I, XI and XIII$_b$ follows the truth of lemma XV in the case (ii).

**Lemma XVI.** If the sign of negation does not occur in the antecedent of the simple expression $x$ and if $lab$ is the consequent of $x$ then $x$ is either a thesis of $AL$ or a rejected expression.

**Proof.** The antecedent of $x$ must fulfil one of the following seven conditions:

(i) either lab or lba is one of its arguments;

(ii) it connects the variable $a$ with the variable $b$ or the variable $b$ with the variable $a$;

(iii) it connects the variable $c$ with the variable $a$ and lba or Icb is its argument;

(iv) it connects the variable $c$ with the variable $b$ and Iac or Ica is its argument;

(v) it connects $c$ with both $a$ and $b$;
(vi) it connects c with a, d with b and either lca or lde is its argument;
(vii) none of the conditions (i) — (vi) is fulfilled by the antecedent of x.

Let us note that our lemma follows:
in the case (i) from lemmas IVa, Xa and Xb;
in the case (ii) from lemmas VIIIa, XIa, and XIV;
in the case (iii) from lemmas VI, Xa, Xb and XIV;
in the case (iv) from lemmas VI, Xa, and XIV;
in the case (v) from lemmas I, XI and XIV;
in the case (vi) from lemmas VI, Xa, Xb and XIV.

We pass in turn to case (vii).

Let us replace in x every variable connected by the antecedent of x with the variable a or b, by a or b, respectively; the remaining variables occurring in x, excepting a and b, let us replace by c. The expression obtained thus from x let us call x'.

Such a substitution is always practicable since in the case vii no variable is connected by the antecedent of x with both a and b.

By way of argument similar to the one used in the proof of lemma XV it is easy to show that none of the expressions

\[ a, b, w, u, \text{ and } v \]

does occur in the antecedent of x'.

We shall prove now that the same is true with respect to:

\[ v, w, u, \text{ and } v \]

It follows from the assumptions of case (vii) that neither lca nor lde does occur in the antecedent of x.

We will show at present that none of these expressions is a result of substitution upon a particular-affirmative expression a which is an argument of the antecedent of x. This would be possible only if either one of the variables occurring in a had the form of one of the variables occurring in the consequent of x and the second were connected by the antecedent of x with the remaining variable of this consequent, or if every variable occurring in a were connected by the antecedent of x with a corre-

responder variable of the consequent of x. Both these possibilities, however, are excluded by the assumptions of case (vii).

The following expressions only may therefore occur in the antecedent of x':

\[ a, b, c, a, b, c, a, b, c, \text{ and } c, b, c, a, b, c, \text{ and } c, b, c, \]

From this and from the lemmas V, VI and X, it results that one of the expressions:

\[ v, w, u, w, u, \text{ and } v \]

is a consequence of x' and hence of x.

From this, from the lemmas I, XIIIa, and from the fact that the expression (6) referred to in definition V is a rejected expression follows the truth of lemma XVI in the case (vi). It has thus been proved in all possible cases.

Lemma XVII. If the sign of negation does not occur in the antecedent of the simple expression x and if the consequent of x is a negation of a general- or particular-affirmative expression then x is a rejected expression.

Proof. Let us substitute the variable a for all the variables occurring in x. Let x' be the expression obtained thus from x. Then, as follows from lemmas V and X, either

\[ N, a \text{ or } N, a \]

is a consequence of x'.

Hence by lemmas I and XIIIa follows lemma XVII.

Theorem II. If x is either an elementary expression or a simple expression whose antecedent does not contain the sign of negation then x is either a thesis of AL or a rejected expression.

Proof. This theorem follows from lemmas VII, XI, XIIa, XIIIa, XV, XVI and XVII.

§ 7. We turn now to simple expressions containing in their antecedent the sign of negation. Let us introduce first the following definition.
Definition XIII. For any natural \( n > 2 \) a meaningful expression of AL \( x \) will be regarded as having the property \( W_n \) if \( x \) is not a thesis of AL and if at the same time it fulfills the following conditions:

1. \( x \) contains at least \( n \) inequiform variables, which means that each of them differs in form from all the remaining;

2. every result of some substitution upon \( x \) which contains less than \( n \) inequiform variables, is a thesis of AL.

Lemma XVIII. For every natural \( n > 2 \) every meaningful expression of AL which is a 1-range consequence of a sequence of meaningful expressions of AL every member of which is either a thesis of AL or has the property \( W_n \), is also either a thesis of AL or has the property \( W_n \).

Proof. Let \( x \) and \( y \) be any meaningful expressions of AL.

It is easy to see that in the case when \( x \) is a result of some substitution upon a meaningful expression of AL which is either a thesis of AL or has property \( W_n \), the expression \( x \) itself is either a thesis of AL or has the property \( W_n \). In this case our lemma is true.

Now we shall prove that expressions obtained from such meaningful expressions of AL which are either theses of AL or posses the property \( W_n \) by means of applying once the rule of implication are also either theses of AL or have the property \( W_n \).

Let us assume first of all that the expression \( y \) does contain less than \( n \) inequiform variables and that both the conditional sentence \( C^xxy \) and its antecedent are either theses of AL or have the property \( W_n \). We shall prove that under these assumptions \( y \) must be a thesis of AL.

For this purpose let us substitute in \( x \) for every variable differing in form from all the variables occurring in \( y \) a variable having the form of an arbitrary variable of \( y \). Let \( x' \) be the expression obtained thus from \( x \).

It is easy to see that \( C^xxy \) is obtainable from \( C^xxy \) by applying substitution and that both \( C^xxy \) and \( x' \) are theses of AL. From this it follows that \( y \) is a thesis of AL.

Next let us assume that \( y \) does contain at least \( n \) inequiform variables and, as above, that \( C^xxy \) and \( x \) are theses of AL or have the property \( W_n \). Let, further, \( y' \) be any result of substitution upon \( y \) containing less than \( n \) inequiform variables.

We shall prove that \( y' \) is a thesis of AL.

Without affecting the generality of our proof we may assume that in carrying through the substitution of which \( y' \) is the result the variable \( a \) occurring in \( y \) has been replaced by \( b \).

Let us substitute in \( x \) for the variable \( a \) if it does occur there the variable \( b \).

Let us adopt an analogous procedure in respect of all variables which occur in \( x \) and whose form is equal to that of some variable occurring in \( y \).

For all the variables occurring in \( x \) not equiform to any variables contained in \( y \) let us substitute variables having the form of an arbitrarily chosen variable of \( y' \).

Let \( x' \) be the expression obtained in this way from \( x \).

Again as above we deduce easily that \( C^xxy' \) is a result of some substitution upon \( C^xy \), that both \( C^xxy' \) and \( x' \) are theses of AL and that hence \( y' \) must also be a thesis of AL. From this it follows that \( y \) is either a thesis of AL or has the property \( W_n \). Thus lemma XVIII has been proved for all possible cases.

Lemma XIX. For every natural \( n > 2 \) every meaningful expression of AL which is a consequence of \( (1) - (4) \) and of any expression having the property \( W_n \), is either a thesis of AL or has the property \( W_n \).

Proof. This lemma follows easily by induction from Lemma XVIII.

Lemma XX. Given any natural number \( n > 2 \), there exists an expression having the property \( W_n \).

Proof. Let \( n \) be any natural number greater than 2 and let

\[
d_1, a_2, \ldots, a_n
\]

be \( n \) inequiform variables. For every two natural numbers \( i \) and \( j \) fulfilling the conditions

\[
1 < i < n, \quad 1 < j < n \quad \text{and} \quad i < j
\]

let us form the expression

\[
\sum_{1}^{n} a_i
\]
Let $x$ be a simple conjunction having as arguments those and only those expressions belonging to (12) which are different from $N^*a_{n-1}0_a$.

We shall prove that the simple expression

$$C^*x1^*a_{n-1}a_n$$

has the property $W_n$.

It follows from IV$_{1,s}$ and $X_n$ that every expression which is a result of substitution upon (13) and contains less than $n$ inequiform variables is a thesis of AL.

It remains to be shown that (13) is not a thesis of AL. For this purpose it suffices to note that the interpretation presented in the proof of theorem I when applied to (13) changes it into a false expression.

Thus we have shown that (13) has the property $W_n$ and hence that lemma XX is true.

**Theorem III.** For every finite sequence of meaningful expressions of AL

$$x_1, x_2, \ldots, x_k$$

none of which is a thesis of AL, there exists a simple expression which neither is a thesis of AL nor a rejected expression with respect to this sequence.

**Proof.** Let $n^i$ be the number of inequiform variables occurring in $x_i$ where $i = 1,2,\ldots,k$. There is a natural number $n$ greater than any of the numbers $n_1, n_2, \ldots, n_k$.

Let $x$ be any expression having the property $W_n$. The existence of such an expression is secured by lemma XX.

Hence by lemma XIX we may infer that every consequence of (1) — (4) and of $x$ which is not a thesis of AL contains at least $n$ inequiform variables. The expression $x$ is therefore not a rejected expression with respect to (14). But $x$, having the property $W_n$, can not be a thesis of AL. Thus $x$ fulfills the conditions of theorem III.

Theorem III answers negatively our second problem formulated in § 2: does there exist a finite sequence of false meaningful expressions of AL with respect to which all meaningful expressions of AL which are not theses of AL would be rejected expressions?

This problem is connected with the following two questions:

1. are all true meaningful expressions of AL theses of AL?
2. does there exist a finite sequence of false meaningful expressions of AL with respect to which every false meaningful expression of AL would be a rejected expression?

We are not able to give a precise answer to these questions for — as mentioned in § 2. — we have no criterion of the truth of meaningful expressions of AL at our disposal.

However, the intuitive meaning of the considerations contained in § 7 suggests an answer to question 2 while leaving questions 1 unanswered.

For in order to prove the existence of the expression $x$ (referred to in the proof of theorem III), which is neither a thesis of AL nor a rejected expression with respect to any set of meaningful expressions of AL, we had to recur to expression (13), occurring in the proof of lemma XX. But this expression seems to be intuitively false.

For this reason we shall aim at strengthening the notion of rejected expression while leaving the notion of thesis of AL unchanged.

Let us note too that assuming the falsity of the expression (13) we may infer from theorem III that no finite set of meaningful expressions of AL is a set of rejected axioms.

§ 8. **Definition XIV.** Let $k$ be any natural number, $a_1$ and $a_2$ — any general or particular-affirmative expressions, and $x$ — any elementary or simple expression; then by a $k$-range rejected expression we shall mean:

in the case $k = 1$ any meaningful expression of AL which is a rejected expression according to definition V, and also any expression having the form,

$$C^*K^*N^*a_1N^*a_2x$$

provided the expressions

$$C^*N^*a_2x \text{ and } C^*N^*a_2x$$

are rejected according to definition V;
in the case \( k > 1 \), any meaningful expression which is rejected with respect to any set of meaningful expressions every member of which is a rejected expression of a lower range than \( k \), and also any expression having the form (15), provided the expressions (16) are rejected expressions of ranges lower than \( k \).

In order to avoid ambiguities, meaningful expressions of AL called hitherto in accord with definition V rejected expressions will be called henceforth rejected expressions according to definition V. Any meaningful expression of AL which for some natural \( k \) is a \( k \)-range rejected expression will be called a rejected expression according to definition XIV.

Definition XIV is a generalisation of definition V. A similar generalisation of definition IV will not be needed.

**Lemma XXI.** Every expression which is rejected according to definition V is rejected according to definition XIV.

**Proof.** This is an immediate conclusion from definition XIV.

**Lemma XXII.** Let \( \alpha_1 \) and \( \alpha_2 \) be any general- or particular-affirmative expressions, and \( x \) any elementary or simple expression.

This being assumed, if one or the expressions (16) is a thesis of AL or if both are rejected expressions according to definition XIV, then every single expression whose consequent is equiform to \( x \), in case of \( x \) being an elementary expression, or to the consequent of \( x \), in case of \( x \) being a simple expression, and whose antecedent has as arguments elementary expressions equiform to \( N^* \alpha_1 \) and \( N^* \alpha_2 \) and to those and only those elementary expressions which are arguments of the antecedent of \( x \) is a thesis of AL or a rejected expression according to definition XIV, respectively.

**Proof.** By lemmas III, IV and definition XIV.

**Lemma XXIII.** Every meaningful expression of AL which is inferentially equivalent to a finite sequence of meaningful expressions each of which is either a thesis of AL or rejected according to definition XIV, is either a thesis of AL or rejected according to definition XIV.

**Proof.** By lemma I and definitions II, III, IV, V, VI and XIV.

The following theorem corresponds to theorem I proved in § 2.
Further, let us assume that the second of the expressions (16) after substituting for the variables \( a_i \), the classes \( A'_i \) \((i = 1, 2, \ldots, n)\), respectively, changes also into a false sentence. Then the expressions \( \alpha_i \) and \( \beta_i \) turn into false sentences, and the expression \( \beta_a \) into a true one. We mark these expressions by \( \alpha''_A \), \( \alpha''_B \), \( \beta''_A \) and \( \beta''_B \), respectively.

For every natural \( i = 1, 2, \ldots, n \) let us define the class \( A''_i \) of all ordered pairs of which the first member is any member of the class \( A'_i \), the second — any member of the class \( A'_j \).

Let us substitute in (15) for the variables \( a_i, a_j, \ldots, a_n \) the classes \( A''_i, A''_j, \ldots, A''_n \), respectively. The expressions obtained from \( \alpha_i, \alpha_j, \beta_i, \) and \( \beta_j \) in result of this substitution let us denote by \( \alpha''_A, \alpha''_B, \beta''_A \) and \( \beta''_B \), respectively.

Let us consider now any four non-empty classes \( X, Y, X_1 \) and \( Y_1 \). Let \( Z \) be the class of all ordered pairs of which the first member is any member of \( X \), the second — any member of \( Y \). Likewise let \( Z_1 \) be the class of all ordered pairs of which the first member is any member of \( X_1 \), the second — any member of \( Y_1 \). It is easy to see that:

- if \( X \) is included in \( X_1 \) and \( Y \) in \( Y_1 \) then \( Z \) is included in \( Z_1 \);
- if \( X \) and \( X_1 \) have a common member, and \( Y \) and \( Y_1 \) have a common member then \( Z \) and \( Z_1 \) have also a common member;
- if it is not true that \( X \) and \( X_1 \) have a common member or that \( Y \) and \( Y_1 \) have a common member, then it is not true that \( Z \) and \( Z_1 \) have a common member;
- if it is not true that \( X \) is included in \( X_1 \) or that \( Y \) is included in \( Y_1 \), then it is not true that \( Z \) is included in \( Z_1 \).

From this it may be inferred easily that if the sentences \( N^*a'_1 \) and \( N^*a'_2 \) are true then also the sentences \( N^*a''_1 \) and \( N^*a''_2 \) are true. And if the sentences \( \beta'_1 \) and \( \beta'_2 \) are false then also the sentence \( \beta''_1 \) is false. And lastly, if \( \beta'_1 \) and \( \beta'_2 \) are true then also \( \beta''_1 \) is true.

In case of a given variable not occurring in a given expression, by a substitute of this expression obtained as the result of replacing this variable by some other expression we understand any expression equiform to the expression in which the replacement is supposed to have been carried out.

Hence it follows that if the interpretation considered here makes the expressions (16) false it changes also the expression (15) into a false expression.

Finally, every expression rejected according to definition \( V \) will be false in this interpretation, since the expression (6) referred to in definition \( V \) is false in it.

From this it is easy to infer that every expression rejected according to definition XIV is made false by the interpretation in question. Hence follows theorem IV.

§ 9. Lemma XXIV. Every simple expression containing signs of negation in its antecedent is either a thesis of \( AL \) or a rejected expression according to definition XIV.

Proof. Let \( k \) be any natural number, \( x \) — any simple expression in the antecedent of which occur \( k \) signs of negation. We shall apply induction in respect of \( k \).

In the case \( k = 1 \) two possibilities have to be considered:

(i) the sign of negation is the first sign of the consequent of \( x \);

(ii) the sign of negation does not occur in the consequent of \( x \).

In the case (i) it follows from lemma IX that \( x \) is inferentially equivalent to a simple expression of which neither the antecedent nor the consequent contains the sign of negation. From this and from lemmas XXI, XXIII and theorem II follows lemma XXIV in case (i).

In case (ii) we may, on the basis of lemmas VII, XXII, assume that in the antecedent of \( x \) occurs an argument not containing the sign of negation as its first sign. It follows from lemma IX that \( x \) is then inferentially equivalent to a simple expression \( x' \) in which both arguments of the antecedent and the consequent contain the sign of negation as their first sign. Let \( N^*a'_1 \) and \( N^*a'_2 \) be those two arguments. Further, let \( y \) be any simple expression the arguments of whose antecedent are solely such elementary expressions which are equiform to some arguments of the antecedent of \( x' \) but differ in form from both the expressions \( N^*a'_1 \) and \( N^*a'_2 \) and whose consequent is equiform to the consequent of \( x' \).
In each of the expressions

\[(20) \quad C^*N^*a_1 \ldots Y, \quad C^*N^*a_2 \ldots Y\]

the sign of negation, — as may be easily seen — occurs once in the antecedent and once in the consequent. These expressions are therefore, as may be inferred from case (i) and lemma III, either theses of AL or rejected expressions according to definition XIV.

From this and from lemma XXII it follows that also in the case (ii) \(x\) is either a thesis of AL or a rejected expression according to definition XIV.

Let us assume now that our lemma is true for any natural number lower than or equal to a given natural \(k\), and that the sign of negation occurs in the antecedent of \(x\) \(k+1\) times.

Let \(N^*a_1\) and \(N^*a_2\) be any arguments of the antecedent of \(x\) having the sign of negation as their first sign. Further, let \(y\) be any simple expression whose antecedent has as arguments those and only those elementary expressions which are equiform to the corresponding arguments of the antecedent of \(x\) which differ in form from \(N^*a_1\) and from \(N^*a_2\), and whose consequent is equiform to the consequent of \(x\).

Then — as is easy to see — each of the expressions (20) contains in the antecedent at most \(k\) sign of negation. Each then — this follows from our assumption that lemma XXIV is true for such expressions — is either a thesis of AL or a rejected expression according to definition XIV.

From this and from lemma XXII it follows that also the expression \(x\) is either a thesis of AL or rejected according to definition XIV.

Thus we have reached the last stage of the inductive proof of lemma XXIV.

**Theorem V.** Every meaningful expression of AL is either a thesis of AL or a rejected expression according to definition XIV.

**Proof.** By lemmas II, XXI, XXIII, XXIV and theorem II.

Let us consider now the following question: is theorem V a solution of the problem formulated in § 2? The answer to this depends first of all on whether every expression rejected according to definition XIV is a false expression.

This would be the case if the following three sentences were true.

I. The expression (6) referred to in definition V is a false expression;

II. If one of the consequence of a meaningful expression of AL \(x\) and of (1) — (4) is false then the expression \(x\) is also false;

III. If the expressions (16) are both false then the expression (15) is also false.

(The symbols \(a_1\) and \(a_2\) occurring in (15) and (16) represent any general- or particular-affirmative expressions, \(x\) — any elementary or simple expression).

No doubt seems possible with respect to sentences I and II. Let us examine sentence III more closely.

In the first place let us note that the restrictions imposed on the antecedents of the expressions (16) are essential. Should we delate them, sentence III would be obviously false. Once however they are assumed the truth of sentence III may be supported by certain arguments whose conformity with intuition requires to be discussed.

Let us note then that the proof of theorem IV contains a reasoning from which it follows that sentence III may be inferred from the following sentence:

IV. Every meaningful expression of AL is false if it is false in the interpretation used in the proof of theorem IV.

As regards this interpretation let us note that the restriction we introduced in its formulation, namely, that merely non empty classes may be substituted for the variables in meaningful expressions of AL is connected with the fact that several theses of AL, as for example the expression (2) which is used by Lukasiewicz as one of the axioms of Aristotelian syllogistic, turn into false expressions if their variables are replaced by empty names.

\[\ldots\]
Assuming now the truth of sentence III (or IV) and of sentences I and II we may infer from theorem V the following answers to the question asked in § 2.

V. The expression (6) referred to in definition V plays an analogous role to that which is ascribed in § 2 to rejected axioms.

VI. The expressions (1) — (4) which form a sufficient basis for the deduction of all true Aristotelian syllogisms are also axioms of the system of all true meaningful expressions of AL.

§ 10. To conclude, I will present an arithmetical interpretation of meaningful expressions of AL, different from the one used in the proof of theorem IV. Leibniz is the author of this interpretation*

For the variables occurring in meaningful expressions of AL we are allowed to substitute solely pairs of relatively prime integers one of which is positive, the second — negative.

The expression 1ab has the following meaning: the first member of the pair a is divisible by the first member of the pair b, the second member of a — by the second member of b.

The expression lab has the following meaning: the first member of a and the second member of b are relatively prime, and so are the second member of a and the first member of b.

We shall prove that all theses of AL and no other meaningful expressions of AL are true in this interpretation.

Let us note first that in this interpretation the expressions (1) — (4) are true, while the expression (6) referred to in definition V is false.

From this it is easy to draw the conclusion that in order to prove Leibniz's interpretation possessing the required property it suffices to show that the falsity in this interpretation of the expressions (16) entails always the falsity in it of the

* Reference to this interpretation was made by Łukasiewicz in 1937 in a paper he read at one of the sessions of the Polish Logical Society. Łukasiewicz writes in his article quoted in § 1 (Aristotelian syllogistic, p. 296): "In 1679 Leibniz has made a discovery which is very important from the point of view of the history of deductive sciences, which however up till now has not been duly appreciated by anybody: he found an arithmetical interpretation of Aristotelian syllogistic (see L. Couturat, Oeuvres et fragments inédites de Leibniz, Paris 1908, p. 77 ff)".

expression (15) (it is assumed as above that a1 and a2 are any general-or particular-affirmative expressions, x — any elementary or simple expression).

Let a1a2...an be any sequence of integers and let us resolve each member of this sequence into prime factors. Let further

\[ u_1, u_2, \ldots, u_n \]

be the sequence of all those prime factors of which none is equal to any of the remaining ones, and

\[ v_1, v_2, \ldots, v_n \]

any sequence of prime numbers each of which differs from all the remaining:

Lastly let i and j be any two natural numbers fulfilling the conditions:

\[ 1 < i < k, \quad 1 < j < k \]

Let us assume now that \( a_i \) is equal to the product of all prime numbers which occur in (21) and whose indices are the numbers:

\[ l_1, l_2, \ldots, l_m \]

and that \( a_j \) is equal to the product of all prime numbers which occur in (21) and whose indices are the numbers

\[ j_1, j_2, \ldots, j_n \]

Let \( b_1 \) and \( b_2 \) be the respective numbers equal to products of prime numbers occurring in (22) and having as their indices the respective sequences (23) and (24).

It is easy to see that \( b_1 \) and \( b_2 \) are relatively prime if and only if \( a_i \) and \( a_j \) are relatively prime, and that \( b_1 \) is a divisor of \( b_2 \) if and only if \( a_i \) is a divisor of \( a_j \).

Let us suppose now that the expressions (16), whose terms are given meanings concordant with Leibniz's interpretation, change into false sentences in the case of substituting of certain pairs of integers for their variables.
In view of the above stated connexions between \( a, b, c, d, e, f \) and \( g, h, i \), we are allowed to assume that the integers occurring in pairs which we have substituted for the variables of the first of the expressions (16) and the integers which we have substituted for the variables of the second of the expressions (16), have no common prime divisors.

Let us assume that the variable \( a \) occurs in the expression (15).

If this variable occurs only in one of the expressions (16) let us substitute for \( a \) in (15) the same pair which we have substituted for \( a \) in that of the expressions (16) which contains \( a \).

If, however, both the expressions (16) do contain the variable \( a \) let us substitute for \( a \) in (15) a pair of which the first member is equal to the product of the first members, the second — to the product of the second members of the pairs substituted for \( a \) in the expressions (16). For the remaining variables occurring in (15) let us substitute pairs of numbers defined analogously as the one above.

In reasoning as in the proof of theorem IV we may show that the expression obtained from (15) in result of the substitution described above is a false sentence.

From this it follows — and thus we reach the end our proof — that all such and only such meaningful expressions of AL which are theses of AL are true in Leibniz's interpretation.

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CANONIC AXIOMATIC SYSTEMS

I intend to precise and develop in this article certain reasonings conveyed generally in connection with the so called Löwenheim — Skolem paradox.

I employ here the issues and terminology from my former publications (16) *). The bibliography is given at the end of this article.

*) In these papers I have given a description of elementary axiomatic systems, and have discerned from among them finitistic elementary systems which may be characterized by the following conditions.

1. A finitistic system is an elementary system, hence a system with a specified syntactical structure and it contains solely nominal variables of one or of finitely many types.

2. The vocabulary of a finitistic system contains a finite number of words from which is build every sentence of the system.

3. The set of axioms of a finitistic system is finite and contains only one analytic axiom which has the following form:

\[ \forall \eta \forall \theta \theta = \eta \]

or a finite number of axioms — one axiom corresponding to each type of variables — having the following form:

\[ \forall \eta_\alpha \exists \theta_\alpha, \theta_\alpha = \eta_\alpha \]

4. The set of rules of inference of a finitistic system is finite and contains: a) the logical rules of inference, to which may be reduced nearly the whole logical calculus, b) nominal definitions i.e. rules permitting the replacement of expressions by their abbreviations and c) the principles of complete induction which are rules of inference corresponding — according to our theory of generalised real definitions — to contextual definitions of sentence-creating functions with nominal arguments.

These elementary finitistic systems in which nominal definitions do not occur are called basic finitistic systems. Meaningful expressions of these systems belong to one of the following two groups: a) logical terms