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PROOF OF Ł-DECIDABILITY OF LEWIS SYSTEM S5[1]

The notion of Ł-decidability is closely related to the notion of rejected proposition introduced by Jan Łukasiewicz. We denote by T_S the set of all theses of an arbitrary deductive system S, by T_S^{-1} — the set of all rejected propositions of this system, by W_S — the set of all meaningful expressions.

A system S is *Ł-decidable* if and only if

1°. $T_S \cup T_S^{-1} = W_S$,

2°. $T_S \cap T_S^{-1} = \emptyset$.

So, every meaningful expression of an Ł-decidable system is either its thesis or a rejected expression and it cannot be a thesis and a rejected expression simultaneously.

Łukasiewicz had used the notion of Ł-decidability in his papers [5]–[8]. This notion is treated of in detail in the papers [1], [9] and [10]. In the last one there has been introduced, too, the term "Ł-decidable".

Łukasiewicz had formulated two rules of rejection which will be denoted by r_1^{-1} and r_2^{-1}. The schemes of these rules have the forms:

$$\mathrm{r}_1^{-1}\ \frac{C\alpha\beta \in T_S,\ \beta \in T_S^{-1}}{\alpha \in T_S^{-1}} \qquad \mathrm{r}_2^{-1}\ \frac{\alpha^* \in T_S^{-1},\ \alpha^* \in \mathrm{Sub}(\alpha)}{\alpha \in T_S^{-1}}$$

where $\mathrm{Sub}(\alpha)$ denotes the set of all expressions obtained from α by substitutions.

The paper [1] contains the proof of the theorem[2]:

THEOREM 1. *The classical calculus of propositions in which the only rules of rejection are* r_1^{-1} *and* r_2^{-1} *and the only rejected axiom is a single variable, e.g. the variable "p", is an Ł-decidable system.*

This paper is the continuation of the papers published in Scientifical Notes of School of Pedagogics in Opole. The paper [1] of Grzegorz Bryll and Marian Maduch contains the proof that all n-valued ($n = 3, 4, \ldots$) implicative, implicative-negative and

[1] The paper has been presented at the conference "Logical Calculi" organized by Section of Logic at the Institute of Philosophy and Sociology of Polish Academy of Sciences in Warsaw, October 1971.

[2] This theorem is given without a proof in the paper [6], p. 109.

definitionally complete logics of Łukasiewicz are Ł-decidable if their only rules of rejection are r_1^{-1} and r_2^{-1} and their only axiom of rejection is the expression

$$C\,\underbrace{Cp\,\ldots\,Cp}_{n-1}q\;\underbrace{Cp\,\ldots\,Cp}_{n-2}q$$

The paper [2] of Grzegorz Bryll and Maria Rosiek contains the proof of the theorem on Ł-decidability of a system of modal logic presented in the paper [8]; the former paper contains the formulation of the theorem only.

Marian Maduch's paper [9] contains the proof that among the systems in which the only rules of rejection are the rules r_1^{-1} and r_2^{-1} there exists a finite system of rejected axioms for these and only these systems for which there does not exist a directed basis.[3] It is also proved in this paper that there does not exist a directed basis for the intuitionistic calculus of propositions. In Andrzej Gniazdowski's paper [4] it is proved that directed bases do not exist, too, for infinite-valued implicative-negative logic of Łukasiewicz and positive logic. Thus, none of the mentioned systems has a complete, finite system of rejected axioms if their only rules cf rejection are the rules r_1^{-1} and r_2^{-1}.

This paper is concerned with Lewis system S5 in which the primitive terms are the functors of the classical implication, negation and necessity. The system is based upon the following axioms, given by Kurt Gödel:

1. *LCCpqCCqrCpr*
2. *LCpCNpq*
3. *LCCNppp*
4. *LCLpp*
5. *LCLCpqCLpLq*
6. *LCLpLLp*
7. *LCNLpLNLp*[4]
8. *CLpp*

The only rules of the system are: the rule of detachment for implication and the rule of substitution. The variant of Lewis system S5 defined in this manner will be denoted by L. The sets W_L and T_L are the sets of all meaningful expressions of the system L and of all its theses, respectively. The variables

$$\alpha, \beta, \gamma, \ldots, \quad \alpha_1, \alpha_2, \ldots, \quad \beta_1, \beta_2, \ldots, \quad \gamma_1, \gamma_2, \ldots$$

are assumed to run over the set of names of expressions of the set W_L. Instead of $\alpha \in T_L$ we shall often write $\vdash \alpha$.

The axioms 1–3 and 8 immediately imply Łukasiewicz's axioms of the classical calculus of propositions. Hence and because the system L is consistent it follows that

THEOREM 2. *If α does not contain the symbol L then $\alpha \in T_L$ if and only if α is a thesis of the classical calculus of propositions.*

[3] We shall treat of this theorem in detail in the section 1.

[4] Let us mention that Dr. Ewa Żarnecka-Biały has shown at the Seminar in Opole that the axioms 6 and 7 may be replaced by the expression *LCMLNLpLp*.

We recall the definition of Wajsberg's matrix[5] which will be denoted by the symbol $\mathfrak{M}_L$.

Let **1** be the infinite sequence with all the elements equal to 1, **0** — the infinite sequence with all the elements equal to 0, U — the set of all infinite sequences with elements equal to 0 or 1. The matrix $\mathfrak{M}_L$ is the ordered system

$$\langle U, \{\mathbf{1}\}, \rightarrow, \sim, \Box\rangle ,$$

where for all $\{a_i\}, \{b_i\} \in U$:

1. $\{c_i\} = \{a_i\} \rightarrow \{b_i\}$ if and only if for every positive integer $n : c_n = 0$ if and only if $a_n = 1$ and $b_n = 0$;
2. $\{d_i\} = \sim \{a_i\}$ if and only if for every positive integer $n : d_n = 0$ if and only if $a_n = 1$;
3. $\Box \{a_i\} = \begin{cases} \mathbf{1} & \text{if} \quad \{a_i\} = \mathbf{1} , \\ \mathbf{0} & \text{if} \quad \{a_i\} \neq \mathbf{1} . \end{cases}$

The following theorem given in the paper [11] is fundamental in investigations concerning Lewis system S5[6].

THEOREM 3. $T_L = E(\mathfrak{M}_L)$.

DEFINITION 1. An expression α is *equivalent* to expressions $\alpha_1, \ldots, \alpha_n$; if and only if

$$\vdash C\alpha\alpha_1, \ldots, \vdash C\alpha\alpha_n, \vdash C\alpha_1 \ldots C\alpha_n \alpha .$$

The equivalence of an expression α and expressions $\alpha_1, \ldots, \alpha_n$ is denoted by $\alpha \sim \alpha, \ldots, \alpha_n$.

DEFINITION 2. An expression has *normal form* if it is of the form

$$(*) \qquad CL\alpha_1 \ldots CL\alpha_m CNL\beta_1 \ldots CNL\beta_n \gamma , \qquad m, n = 0, 1, \ldots$$

and expressions $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n, \gamma$ do not contain the symbol L.

The expression $(*)$ will be denoted by φ from now on. Let us introduce further symbols:

$K(\alpha_1) = \alpha_1 ,$
$K(\alpha_1, \ldots, \alpha_n) = NC\alpha_1 \ldots C\alpha_{n-1} N\alpha_n;$
$A(\alpha_1) = \alpha_1 ,$
$A(\alpha_1, \ldots, \alpha_n) = CN\alpha_1 \ldots CN\alpha_{n-1} \alpha_n;$
$\varepsilon_0(\varphi) = CK(\alpha_1, \ldots, \alpha_m) \gamma ,$
$\varepsilon_i(\varphi) = CK(\alpha_1, \ldots, \alpha_m) \beta_i, \qquad i = 1, \ldots, n .$

It is essential in further considerations that the symbol L does not appear in the expressions $\varepsilon_i(\varphi)$ $(i = 0, \ldots, n)$.

Proofs of two following theorems are given in the paper [3].

THEOREM 4. *For every* α *there exist expressions* $\alpha_1, \ldots, \alpha_n$ *having normal form such that* $\alpha \sim \alpha_1, \ldots, \alpha_n$.

[5] Cf. the paper [11].
[6] A certain proof of this theorem is also given in the paper [3].

THEOREM 5. *If for a certain positive integer i ($0 \leqslant i \leqslant n$) the expression $\varepsilon_i(\varphi)$ is a thesis of the classical calculus of propositions then the expression φ is a thesis of the system* L.

In the section 1 we shall prove that for the system L there exists a directed basis. In the section 2 we shall formulate the rule of rejection r_3^{-1} and show that if the rules of rejection r_1^{-1}–r_3^{-1} hold in the system L and its only rejected axiom is a single variable then this system is Ł-decidable.

§ 1. Let S be an arbitrary system of calculus of propositions the language of which contains the functor of implication and in which the rules are: the rule of detachment for implication and the rule of substitution. Denote by W_S the set of all meaningful expressions of this system, by T_S — the set of all its theses. Let $X \subseteq W_S$. The expression $\mathrm{Cn}_S(X)$ denotes the smallest set containing the set X and closed with respect to the rules of substitution and detachment. We shall employ the introduced notation in the formulations of the definition 3 and the theorem 6.

DEFINITION 3. A family of sets $\mathbf{R}$ is a *directed basis* of the system S if and only if

1°. $\bigwedge\limits_{X \in \mathbf{R}} (T_S \neq X \subseteq W_S \wedge \mathrm{Cn}_S(X) = X)$,

2°. $\bigcap\limits_{X \in \mathbf{R}} X = T_S$,

3°. $\bigwedge\limits_{X, Y \in \mathbf{R}} \; \bigvee\limits_{Z \in \mathbf{R}} Z \subseteq X \cup Y$.

By a complete system of rejected axioms of the system S with fixed rules of rejection we shall mean every set $X \subseteq W_S$ such that the set of all expressions rejected on the grounds of the set X and adopted rules is identical with the set $W_S - T_S$.

THEOREM 6. *For the system S in which the only rules of rejection are the rules r_1^{-1} and r_2^{-1} there does not exist a complete finite system of rejected axioms if and only if there exists a directed basis for this system.*

In our further considerations we shall need the infinite sequence of matrices $\mathfrak{M}_2$, $\mathfrak{M}_3$, ...

Let $\mathbf{1}_k$ be the k-element sequence with all the elements equal to 1, $\mathbf{0}_k$ — the k-element sequence with all the elements equal to 0, U_k — the set of all k-element sequences with elements equal to 0 or 1. The matrix $\mathfrak{M}_k (k \geqslant 2)$ is the ordered system

$$\langle U_k, \{\mathbf{1}_k\}, \underset{k}{\rightarrow}, \underset{k}{\sim}, \underset{k}{\Box} \rangle,$$

where for all $\{a_i\}, \{b_i\} \in U_k$:

1. $\{c_i\} = \{a_i\} \underset{k}{\rightarrow} \{b_i\}$ if and only if for every positive integer $n \leqslant k$: $c_n = 0$ if and only if $a_n = 1$ and $b_n = 0$;

2. $\{d_i\} = \underset{k}{\sim} \{a_i\}$ if and only if for every positive integer $n \leqslant k : d_n = 0$ if and only if $a_n = 1$;

3. $k \{a_i\} = \begin{cases} \mathbf{1}_k & \text{if} \quad \{a_i\} = \mathbf{1}_k, \\ \mathbf{0}_k & \text{if} \quad \{a_i\} \neq \mathbf{1}_k. \end{cases}$

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[7] Cf. the paper [4].

[8] Cf. the paper [9].

It is easy to verify that the following formulae are true:

$$\ldots \subsetneqq E(\mathfrak{M}_3) \subsetneqq E(\mathfrak{M}_2)$$

$$\bigcap_{i=2}^{\infty} E(\mathfrak{M}_i) = E(\mathfrak{M}_L)\,.$$

Hence, from the definition 3 and the theorem 8 it follows that the set

$$\{X: \bigvee_{i \geqslant 2} X = E(\mathfrak{M}_i)\}$$

is a directed basis of the system L. This remark and the theorem 6 imply

THEOREM 7. *If the only rules of rejection for the system* L *are the rules* r_1^{-1} *and* r_2^{-1} *then there does not exist a complete finite system of rejected axioms for the system* L.

§ 2. We give the definition of the rule of rejection which will be denoted by r_3^{-1}.

We assume that the symbol L does not appear in the expressions $\alpha, \beta_1, \ldots, \beta_n$ and that $n \geqslant 1$. If all the expressions

$$C\alpha\beta_1, \ldots, C\alpha\beta_n$$

are rejected then the expression

$$CL\alpha A\,(L\beta_1, \ldots, L\beta_n)$$

is also rejected.

DEFINITION 4. $\alpha \in T_L^{-1}$ if and only if α is an expression rejected on the grounds of the rules r_1^{-1}–r_3^{-1} and on the ground of the rejected axiom which is a single variable.

We shall often write $\dashv \alpha$ instead of $\alpha \in T_L^{-1}$. The scheme of the rule r_3^{-1} may be recorded as follows:

$$r_3^{-1}\ \frac{\dashv C\alpha\beta_1, \ldots, \dashv C\alpha\beta_n}{\dashv CL\alpha A\,(L\beta_1, \ldots, L\beta_n)} \qquad n \geqslant 1$$

THEOREM 8. *If* $\dashv \varepsilon_0(\varphi), \ldots, \dashv \varepsilon_n(\varphi)$ *then* $\dashv \varphi$.

Proof. It follows from the assumption of the theorem and from the notation introduced at the beginning of the paper that

$$\dashv CK(\alpha_1, \ldots, \alpha_m)\,\gamma\,,$$

$$\dashv CK(\alpha_1, \ldots, \alpha_m)\,\beta_i\,, \qquad i = 1, \ldots, n\,.$$

So, according to the rule r_3^{-1}:

$$(1)\ \dashv CLK(\alpha_1, \ldots, \alpha_m)\,A\,(L\gamma, L\beta_1, \ldots, L\beta_n)\,.$$

It is easy to check that the expression

$$C\varphi CLK(\alpha_1, \ldots, \alpha_m)\,A\,(L\gamma, L\beta_1, \ldots, L\beta_n)$$

belongs to the set $E(\mathfrak{M}_L)$, and therefore — in view of the theorem 3 — to the set T_L, Hence, from the formula (1) and in virtue of the fact that the rule r_3^{-1} is valid in the system L follows the thesis of the theorem.

THEOREM 9. *If the symbol L does not appear in an expression α and α is a rejected expression of the classical calculus of propositions in which there hold the rules of rejection* r_1^{-1} *and* r_2^{-1} *and in which the only rejected axiom is a single variable then* $\alpha \in T_L^{-1}$.

This theorem is an easy conclusion from the theorem 2, the definition 4 and the definitions of the rules r_1^{-1} and r_2^{-1}.

THEOREM 10. *If the symbol L does not appear in α then* $\alpha \in T_L \cup T_L^{-1}$.

Proof. Suppose that

(1) $$\alpha \notin T_L \cup T_L^{-1}.$$

Hence and from the theorem 2 it follows that α is not a thesis of the classical calculus of propositions. According to the theorem 1 if the rules r_1^{-1} and r_2^{-1} hold in this calculus and the only rejected axiom is a single variable then α is a rejected expression of this calculus. Hence and from the theorem 9 it follows that $\alpha \in T_L^{-1}$. This conclusion contradicts the assumption (1).

The following theorem generalizes the theorem 10.

THEOREM 11. $T_L \cup T_L^{-1} = W_L$.

Proof. Suppose that

(1) $$\alpha \notin T_L \cup T_L^{-1}.$$

The theorem 4 implies existence of expressions $\alpha_1, \ldots, \alpha_n$ having normal form such that

(2) $$\alpha \sim \alpha_1, \ldots, \alpha_n .$$

Hence, from the definition 1 and from the assumption (1) there follows existence of a positive integer i_1 $(1 \leqslant i_1 \leqslant n)$ such that

(3) $$\alpha_{i_1} \notin T_L .$$

Suppose that the expression α_{i_1} is of the same from as the expression φ. It follows from the theorem 5 and the formula (3) that none of the expressions

(4) $$\varepsilon_0(\alpha_{i_1}), \quad \ldots, \quad \varepsilon_n(\alpha_{i_1})$$

is a thesis of the classical calculus of propositions. Hence, from the theorem 10 and because the expressions (4) do not contain the symbol L we obtain the formulae:

$$\dashv \varepsilon_0(\alpha_{i_1}), \ldots, \dashv \varepsilon_n(\alpha_{i_1}) .$$

According to the theorem 8 : $\dashv \alpha_{i_1}$. Hence, from the formula (2), the definition 1 and the remark that the rule r_1^{-1} holds in the system L it follows that $\alpha \in T_L^{-1}$. This conclusion contradicts the assumption (1).

In order to show the system L to be Ł-decidable it remains to prove

THEOREM 12. $T_L \cap T_L^{-1} = \varnothing$.

Proof. It is easy to check that any expression belonging to the set T_L^{-1} does not belong to the set $E(\mathfrak{M}_L)$. This and the theorem 3 imply validity of the theorem we prove.

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