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WALENTY STASZEK

ON PROOFS OF REJECTION

1. INTRODUCTION

The notion of rejected proposition was introduced to logic by Jan Łukasiewicz in connection with his inquiries about Aristotle's syllogistics. Łukasiewicz distinguishes the propositions which are axiomatically rejected from the ones rejected on the ground of "rules of rejection". Here are these rules:

a) the rule of rejection by detachment: if an implication is an asserted expression of the considered system and the consequent is rejected expression then the precedent is also rejected expression.

b) the rule of rejection by substitution: if a substitution of a given expression α is a rejected expression then α is also a rejected expression.

Łukasiewicz was using his notion of rejected proposition when he examined systems with no quantifiers. Therefore he did not need any rules of rejection which would have concerned quantifiers. In this paper we shall take into account such rules, too.

Łukasiewicz did not give a definition of the proof resulting in rejection of an expression. To hint how he could understand this notion we shall repeat a proof given in [1], however, we shall use symbols different from the ones of Łukasiewicz.

Łukasiewicz proves that the expression SiP is rejected; he bases his proof on theses of calculus of propositions and the following rejected axiom:

*1. $\mathrm{PaM} \wedge \mathrm{SaM} \rightarrow \mathrm{SiP}$

where the asterisk before the number indicates that the expression is rejected. Then he adds the following thesis of calculus of propositions to the proof:

2. $p \rightarrow (q \rightarrow p)$

The next expression is a substitution of this thesis

3. $\mathrm{SiP} \rightarrow (\mathrm{PaM} \wedge \mathrm{SaM} \rightarrow \mathrm{SiP})$

By means of the rule b) we obtain from the expressions 1 and 3

*4. SiP

In paper [2] I have used another notion of proof of rejection which seems to be simpler and more convenient. The definitions of both notions of the proof will be given in section 2 of this paper and their interrelations will be dealt with in section 3.

2. DEFINITIONS OF PROOFS OF REJECTION

We shall adopt some conventions concerning terminology and symbols and we shall make some assumptions.

By usual proofs we shall mean proofs resulting in assertion of an expression. We do not repeat the definition of this notion, we shall use it in the stock meaning. We do not put, too, any restrictions concerning one-premise rules which are allowed to be used in usual proofs; but we assume that there is the rule of substitution among them and we denote it by RS^+. If we want to indicate that the expression B can be added to the proof on the ground of one of adopted one-premise rules and because an expression A already belongs to the proof then we write $A \vdash B$.

We assume that the only non-one-premise rule which is allowed to be used in usual proofs is the rule of detachement; this assumption is inessential but simplifies considerably enough our deliberations. This rule will be denoted by RD^+. If we want to indicate that the expression A can be added to the proof in virtue of this rule and because expressions B and C belong already to the proof we write

$$BC \vdash A$$

and we assume that $B = \ulcorner C \to A \urcorner$.

The proofs of rejection in the sense of Łukasiewicz will be called Ł-proofs.

The proofs of rejection which I have used in [2] will be called i-proofs. Intuitively, they are close to indirect proofs and that is why the letter "i" appears in their name.

We use rules of rejection in Ł-proofs and there are talked about rules of detachment and substitution among them; they will be denoted by RD^- and RS^- respectively.

We assume that to every one-premise rule which allows to add a new expression to the usual proof there corresponds a dual rule of rejection in Ł-proof. That is, if a given rule which is valid in usual proofs allows to assert an expression B in virtue of the expression A then the dual rule allows to reject the expression A in virtue of the expression B. In this sense the rule of rejection by substitution is dual to the ordinary rule of substitution. If we want to indicate that the expression A can be added to the Ł-proof on the ground of an arbitrary of adopted one-premise rules of rejection and because an expression B appears in the proof as a rejected expression then we write $B \dashv A$.

The adopted agreements imply

Corollary 1. *$A \vdash B$ if and only if $B \dashv A$.*

We assume that the only non-one-premise rule of rejection is the rule of rejection by detachment.

If we want to indicate that the expression C can be added to the Ł-proof on the ground of this rule and because an expression A is rejected and an expression B is asserted then we write $BA \dashv C$, and we assume that $B = \ulcorner C \to A \urcorner$. The adopted notation implies

Corollary 2. *$BC \vdash A$ if and only if $BA \dashv C$.*

Now we shall give more precise definitions of terms which we have been using loosely enough so far. Let

(α) $\quad A_1, A_2, \ldots, A_n$

be an arbitrary sequence of expressions. The element A_i of this sequence will be called an asserted expression if there exists a subsequence of the sequence (α) which is the usual proof of the expression A_i. The expression A_j is called a rejected expression of the sequence (α) if either it is a rejected axiom or for some $k < j$ A_k is not an asserted expression of the sequence (α) and $A_k \dashv A_j$ or there exist $k, l < j$ such that A_k is an asserted expression of the sequence (α) but A_l is not such an expression and $A_k \cdot A_l \dashv A_j$.

In order to indicate that A_i is an asserted expression of the sequence (α) we write $\vdash_\alpha A_i$ and in order to indicate that it is a rejected expression we write ${}_\alpha\dashv A_i$.

Definition 1. The sequence (α) is a Ł-proof of an expression A if

1° for every $i \leqslant n$ $\vdash_\alpha A_i$ or ${}_\alpha\dashv A_i$

and

2° $A_n = A$ and ${}_\alpha\dashv A_n$.

It is seen that the proof taken from Łukasiewicz's monograph [1] is a proof in the sense of the definition 1 — under the assumption that we number theses of calculus of propositions into the set of axioms of Aristotle's syllogistics.

Definition 2. The sequence (α) is an i-proof of an expression A if

1° the sequence (α) is a usual proof

and

2° $A_1 = A$ and A_n is a rejected axiom.

3. INTERRELATIONS AMONG PROOFS OF REJECTION

The following lemma easily follows from the definitions 1 and 2.

Lemma. *Let an expression A be the only element of a sequence. Then the sequence is a Ł-proof of this expression if and only if A is a rejected axiom.*

i-proofs have analogous property.

Theorem 1. *If there exists a Ł-proof of an expression A then there also exists an i-proof of this expression.*

Proof. Let the sequence

(α) $A_1, A_2, \ldots, A_n$

be a Ł-proof of the expression A. According to the definition 1

(1) ${}_\alpha\dashv A_n$ and $A_n = A$

We shall prove the theorem by induction on the length of the sequence (α). For $n = 1$ the theorem is true in virtue of the lemma. Let n be an integer greater than 1. We assume the theorem to be true for every positive integer $l < n$. The lemma easily implies that the theorem is true if A_n is a rejected axiom. So, we can assume that there exist numbers $k, l < n$ for which either

a) $A_l \dashv A_n$

or

b) $A_k A_l \dashv A_n$

Let us consider the case a) first. It is easy to see that ${}_{\alpha}\dashv A_l$ and the sequence

$$A_1, A_2, \ldots, A_l$$

is a Ł-proof of the expression A_l. This and inductive assumption imply existence of a sequence

(β) $B_1, B_2, \ldots, B_i$

which is an i-proof of the expression A_l. According to the definition 2

(2) $B_1 = A_l$

(3) B_i is a rejected axiom.

The assumption that $A_l \dashv A_n$ and corollary 1 imply

$$A_n \vdash A_l$$

This formula, conditions (1)—(3) and the fact that the sequence (β) is an i-proof imply the sequence

$$A\,(= A_n),\ B_1(= A_1),\ B_2, \ldots, B_i$$

to be an i-proof of the expression A.

Suppose now that the condition b) is fulfilled. This and the definition 1 imply

(4) $\vdash_{\alpha} A_k,$ (4a) ${}_{\alpha}\dashv A_l$

Hence, there exists a subsequence of (α)

(γ) $C_1, C_2, \ldots, C_j$

which is a usual proof of the expression A_k, and so

(5) $C_j = A_k$

It follows from (4a) that the sequence $A_1, A_2, \ldots, A_l$ is a Ł-proof of the expression A_l. This and inductive assumption imply existence of a sequence

(δ) $D_1, D_2, \ldots, D_s$

which is an i-proof of the expression A_l, and so we have

(6) $D_1 = A_l$ and

(7) The expression D_s is a rejected axiom.

Let us remark, moreover, that $A_k\, A_l \dashv A_n$ and the corollary 2 imply $A_k\, A_n \vdash A_l$. This, conditions (1), (5) and (6) and remarks that the sequence (γ) is a usual proof and the sequence (δ) is an i-proof bring us to conclusion that the sequence

$$A\,(= A_n),\ C_1, C_2, \ldots, C_j\,(= A_k),\ D_1\,(= A_l), \ldots, D_s$$

is an i-proof of the expression A.

To prove the converse it is necessary to make some assumptions concerning systems on the ground of which the proofs are carried out.

A. *Systems with quantifiers*. We assume that if the quantifiers appear in the system[1] S then the following version of the theorem on deduction is valid in this system:

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[1] We assume also that no operators different from quantifiers appear in the system S and that in the system S the following rule of rejecting expressions (denoted by $R\Pi^{-}$) is valid:

$$\prod_{\alpha} A(\alpha) \dashv A(\alpha)$$

where α is the name of variable of an arbitrarv svntactic category.

THEOREM I. *If the sequence*

(α) $A_1, A_2, \ldots, A_n$

is a usual proof of the expression A_n *and the expression* A_1 *is the only premise in this proof which is different from a substitution of an axiom of the system S and from any logical thesis then the expression*

$$\Pi A_1 \to A_n$$

is a theorem of the system and the expression ΠA *is obtained from* A_1 *by putting universal quantifiers binding all free variables of the expression* A_1 *in front of it.*

THEOREM 2a. *Let there be given a system S with quantifiers in which the theorem* **I** *is true and the rule* $R\Pi^-$ *holds. If an expression A has an i-proof then there also exists its Ł-proof.*

Proof. Let the sequence

(α) $A_1, A_2, \ldots, A_n$

be an i-proof of the expression A. According to the definition 2 the following conditions are fulfilled:

(1) $A_1 = A$

and

(2) the expression A_n is a rejected axiom.

We shall construct a sequence which will be the Ł-proof of the expression A. In virtue of the above assumptions the theorem I is a theorem of the system S. Thus, there exists a sequence

(β) $B_1, B_2, \ldots, B_i$

which is a usual proof of this expression, and so we have

$$B_i = \ulcorner \Pi A_1 \to A_n \urcorner$$

Suppose that the expression ΠA_1 is of the form:

$$\underset{\alpha_1}{\Pi}\ \underset{\alpha_2}{\Pi} \ldots \underset{\alpha_k}{\Pi}\ A_1(\alpha_1, \alpha_2, \ldots, \alpha_k)$$

It follows from the theorem I, conditions (1), (2) and the remark that (β) is a usual proof that the sequence

(φ) $B_1, B_2, \ldots, B_i \; (= \ulcorner \underset{\alpha_1}{\Pi}\ \underset{\alpha_2}{\Pi} \ldots \underset{\alpha_k}{\Pi}\ A_1(\alpha_1, \alpha_2, \ldots, \alpha_k) \to A_n \urcorner), A_n,$

$\underset{\alpha_1}{\Pi}\ \underset{\alpha_2}{\Pi} \ldots \underset{\alpha_k}{\Pi}\ A_1(\alpha_1, \alpha_2, \ldots, \alpha_k), \underset{\alpha_2}{\Pi}\ \underset{\alpha_3}{\Pi} \ldots \underset{\alpha_k}{\Pi}\ A_1(\alpha_1, \alpha_2, \ldots, \alpha_k), \ldots,$

$A_1(\alpha_1, \alpha_2, \ldots, \alpha_k) \; (= A)$

is a Ł-proof of the expression A and

$\vdash_\varphi B_1, \ldots, \vdash_\varphi B_i, {}_\varphi\!\dashv A_n, {}_\varphi\!\dashv \underset{\alpha_1}{\Pi}\ \underset{\alpha_2}{\Pi} \ldots \underset{\alpha_k}{\Pi}\ A_1(\alpha_1, \alpha_2, \ldots, \alpha_k),$

${}_\varphi\!\dashv \underset{\alpha_2}{\Pi}\ \underset{\alpha_3}{\Pi} \ldots \underset{\alpha_k}{\Pi}\ A_1(\alpha_1, \alpha_2, \ldots, \alpha_k), {}_\varphi\!\dashv \underset{\alpha_3}{\Pi}\ \underset{\alpha_4}{\Pi} \ldots \underset{\alpha_k}{\Pi}\ A_1(\alpha_1, \alpha_2, \ldots, \alpha_k),$

$\ldots, {}_\varphi\!\dashv A_1(\alpha_1, \alpha_2, \ldots, \alpha_k)$

B. *Systems without quantifiers*. Let T be an arbitrary system in which the quantifiers do not appear. By conjunctions we shall mean substitutions of expressions of calculus of propositions which are of the type

$$p_1 \wedge (p_2 \wedge (p_3 \wedge \ldots \wedge (p_{n-1} \wedge p_n) \ldots)) \qquad n \geqslant 1$$

Let A be an arbitrary expression of the system T where

(1) $s_1, s_2, \ldots, s_m$

are all its variables[2]. Let

(τ) $t_1, t_2, \ldots, t_n$

be an arbitrary finite sequence of constants and variables of these syntactic categories only which are represented by the variables (1). By

(2) $K_\tau(A)$

we denote the conjunction of all expressions which can be obtained from A by means of substitution of elements of the sequence (τ) for variables (1). We assume that all members of this conjunction are different.

In formulation of the next theorem we preserve introduced notation. We assume that the following version of the theorem on deduction holds in the system T:

THEOREM II. *Let the sequence*

(α) $A_1, A_2, \ldots, A_n$

be a usual proof and A_1 *be the only premise which is different from a substitution of an axiom of the system T and from any logical law. Let also* (1) *be all variables of the expression* A_1 *and* (τ) *— all expressions which have been substituted in the proof* (α). *Then the expression*

$$K_\tau(A_1) \to A_n$$

is a theorem of the system T.

We shall introduce a new one-premise rule of rejection, denoted by RK⁻. According to this rule

$$K^*(A) \dashv A$$

where the expression $K^*(A)$ is of the following form: We distinguish a set of variables

(a) $s_1, s_2, \ldots, s_n$

from the expression A and the expressions

(b) $A^{(1)}, A^{(2)}, \ldots$

each different from the other, are obtained from A by substitution of variables different from (a) for the variables (a) and every time we substitute different variables. By $K^*(A)$ we denote an arbitrary conjunction of expressions taken from (b). The dual rule to this one is secondary rule in all these systems in which the rules of detachment and substitution hold and in which every substitution of the expression

$$p_1 \to (p_2 \to (\ldots \to (p_n \to p_1 \wedge (p_2 \wedge \ldots \wedge (p_{n-1} \wedge p_n) \ldots)$$

is a theorem.

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[2] They can belong to different syntactic categories.

THEOREM 2b. *If no operators appear in the system T, the theorem* II *is true in it and the rule* RK⁻ *holds in then for every expression which has an i-proof there also exists its Ł-proof.*

Proof. We shall use notation we have introduced before. Let the sequence

(α) $A_1, A_2, \ldots, A_n$

be the i-proof of the expression A. Hence

(3) $A_1 = A$

and

(4) A_n is a rejected axiom.

It is easy to see that the sequence (α) fulfills assumptions of the theorem II. So, the expression

(5) $K_\tau(A_1) \rightarrow A_n$

is a theorem of the system T and there exists a sequence

(β) $B_1, B_2, \ldots, B_l$

which is a usual proof of this expression. Let $K_1^*(A_1)$ be an arbitrary expression of the form $K^*(A_1)$ with the condition placed upon it that $K_\tau(A_1)$ is its substitution. One can see that there exists an expression which satisfies these conditions. This and the rule of rejection by substitution imply

(6) $K_\tau(A_1) \dashv K_1^*(A_1)$

We have assumed that the rule RK⁻ holds in the system T, and so

(7) $K_1^*(A_1) \dashv A_1$

It follows from the remark that the sequence (β) is a usual proof of the expression (5), remark (4) and formulae (6), (7) and (3) that the sequence

(φ) $B_1, B_2, \ldots, B_l(=\ulcorner K_\tau(A_1) \rightarrow A_n \urcorner), A_n, K_\tau(A_1), K_1^*(A_1), A_1(=A)$

is a Ł-proof of the expression A, and

$$\vdash_\varphi B_1, \vdash_\varphi B_2, \ldots, \vdash_\varphi B_l, {}_\varphi\!\dashv A_n, {}_\varphi\!\dashv K_\tau(A_1), {}_\varphi\!\dashv K_1^*(A_1), {}_\varphi\!\dashv A_1(=A).$$

BIBLIOGRAPHY

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