# Applying the Inverse Method to Refutation Calculi

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Poznań Reasoning Weak 2018

Refutation Symposium

Poznań, September 15th, 2018

# Outline

- Inverse Method

  Forward proof-search strategy.

- Forward Calculi

  Calculi allowing for terminating forward proof-search.

- Refutation Calculi

  Calculi to formally prove the non-validity of a formula.

**Main Questions**

*Can the Inverse Method meet Refutation ?*

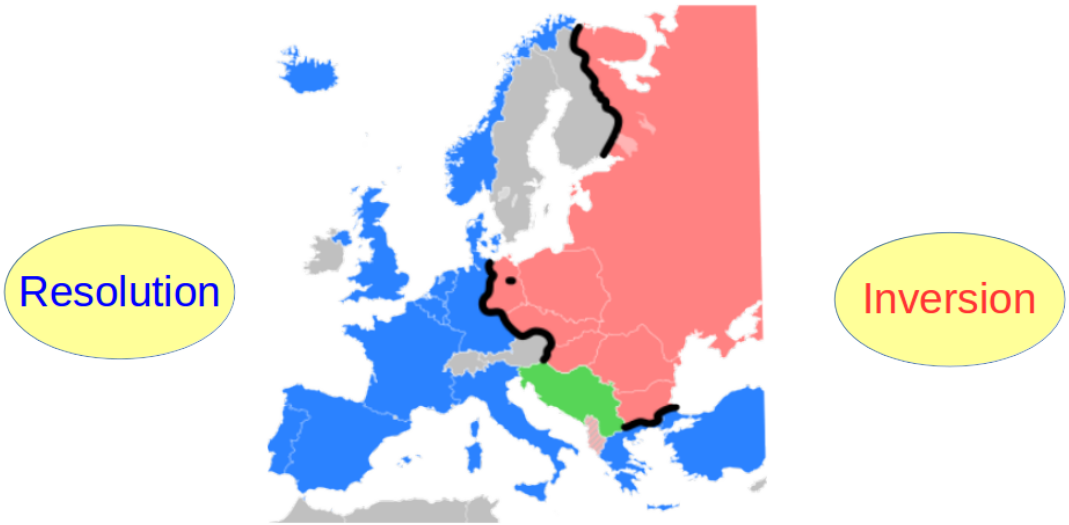
*Can we design Forward Refutation calculi ?*

# Outline

# The Inverse Method

- The *inverse method*, introduced in the 1960s by Maslov, is a saturation based theorem proving technique closely related to (hyper)resolution
- It relies on a *forward* proof-search strategy and can be applied to cut-free calculi enjoying the subformula property.
- Some references:
  - S. Ju. Maslov. An invertible sequential version of the constructive predicate calculus. Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 1967.
  - A. Degtyarev and A. Voronkov. *The inverse method*. Handbook of Automated Reasoning, 2001.

# The Iron Curtain of Automated Reasoning




*A large part of the work on automated reasoning done in the Soviet Union in the sixties and seventies was based on the inverse method proposed by Sergey Maslov in 1964.*

*The role of the inverse method in the Soviet work on proof procedures for predicate logic can be compared to the role of resolution method in theorem proving projects in the West.*

*For a number of reasons, this work has not been duly appreciated outside a small circle of Maslov's associates.*

V. Lifschitz. *What is the inverse method?*. JAR, 1989

# Towards an Inverse Calculus for $\mathrm{CPL}$

We present a forward sequent calculus for $\mathrm{CPL}$ (Classical Propositional Logic), we call **FCL**.

- Sequents have the form

  $\Gamma \vdash \Delta$ $\qquad\qquad$ $\Gamma$, $\Delta$: sets of formulas

- Soundness

  If $\Gamma \vdash \Delta$ is provable in **FCL**, then the sequent $\Gamma \vdash \Delta$ is valid, namely:

  *the formula* $\bigwedge \Gamma \supset \bigvee \Delta$ *is valid in* $\mathrm{CPL}$

- Completeness

  If $G$ is valid in $\mathrm{CPL}$, then the sequent $\vdash G$ is provable in **FCL**

- Proof-search

  **FCL** allows for terminating forward proof-search:

  *start from axioms and apply rules top-down*

# Towards an Inverse Calculus for CPL

- Axioms

  Axioms of **FCL** have the form

  $$p \vdash p \qquad\qquad p\text{: propositional variable}$$

- Rule for $R\vee$ (right or)

  In **FCL** there are two one-premise rules to introduce $A \vee B$ on the right.

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\, R\vee$$

If $\Gamma \vdash A, \Delta$ is valid, then
$\Gamma \vdash A \vee B, \Delta$ is valid

$$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\, R\vee$$

If $\Gamma \vdash B, \Delta$ is valid, then
$\Gamma \vdash A \vee B, \Delta$ is valid

# Towards an Inverse Calculus for CPL

- Rule for $R\wedge$ (right and)

  In **FCL** there is one two-premise rule to introduce $A \wedge B$ on the right.

$$\frac{\Gamma_1 \vdash A, \Delta_1 \qquad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B, \Delta_1, \Delta_2}\, R\wedge$$

If both $\Gamma_1 \vdash A, \Delta_1$ and $\Gamma_2 \vdash B, \Delta_2$ are valid, then $\Gamma_1, \Gamma_2 \vdash A \wedge B, \Delta_1, \Delta_2$ is valid

  Note that all the side formulas in the premises (namely the formulas in the sets $\Gamma_1$, $\Delta_1$, $\Gamma_2$, $\Delta_2$) must be kept in the conclusion.

- The remaining rules for left and right connectives follow the same style.

# Towards an Inverse Calculus for CPL

$$\frac{}{p \vdash p}\ \mathrm{Ax} \qquad \frac{\Gamma, A_k \vdash \Delta}{\Gamma, A_1 \wedge A_2 \vdash \Delta}\ L\wedge \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \qquad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B, \Delta_1, \Delta_2}\ R\wedge$$

$$\frac{\Gamma_1, A \vdash \Delta_1 \qquad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee B \vdash \Delta_1, \Delta_2}\ L\vee \qquad \frac{\Gamma \vdash A_k, \Delta}{\Gamma \vdash A_1 \vee A_2, \Delta}\ R\vee$$

$$\frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \supset B \vdash \Delta_1, \Delta_2}\ L\supset \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A \supset B, \Delta}\ R\supset \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \supset B, \Delta}\ R\supset$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ L\neg \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ R\neg$$

$k \in \{1, 2\}$

# Towards an Inverse Calculus for CPL

## Example

Let us prove the formula $p \vee \neg p$ (goal formula)
We need to build in **FCL** a forward derivation of the sequent $\vdash p \vee \neg p$.

| | | |
|---|---|---|
| (1) | Ax | $p \vdash p$ |
| (2) | $R\neg$ (1) | $\vdash \neg p, p$ |
| (3) | $R\vee$ (2) | $\vdash p \vee \neg p, p$ |
| (4) | $R\vee$ (3) | $\vdash p \vee \neg p, p \vee \neg p$ |
| | $\equiv$ | $\vdash p \vee \neg p$ |

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\, R\neg \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\, R\vee \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\, R\vee$$

# Towards an Inverse Calculus for CPL

- The calculus is sound and complete, but proof-search is non-terminating

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\, R\vee$$

| | | |
|---|---|---|
| (1) | Ax | $p \vdash p$ |
| (2) | $R\vee$ (1) | $p \vdash p \vee p_1$ |
| (3) | $R\vee$ (2) | $p \vdash p \vee p_1 \vee p_2$ |
| (4) | $R\vee$ (3) | $p \vdash p \vee p_1 \vee p_2 \vee p_3$ |
| | $\vdots$ | |

- Solution: exploit the subformula property!

# Towards an Inverse Calculus for CPL

Let $G$ be the formula to be proven (goal formula).

**Subformula property**

To prove $G$, only subformulas of $G$ are needed.

Let $\mathrm{Sf}(G)$ be the set of subformulas of $G$.

We can bound the calculus **FCL** so that the proved sequents only contain formulas in $\mathrm{Sf}(G)$.

- Axioms

$$p \vdash p \qquad p\text{: propositional variable}$$

We require that

$$p \in \mathrm{Sf}(G)$$

# Towards an Inverse Calculus for CPL

- Rule for $R\vee$ (right or)

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\, R\vee \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\, R\vee$$

We can inductively assume that all the formulas in the premise are subformulas of $G$.

We require that the formula introduced in the conclusion is a subformula of $G$:

$$A \vee B \in \mathrm{Sf}(G)$$

- The other rules are fixed accordingly.
- We call the resulting *specialized* calculus **FCL**($G$):
  *Forward Calculus for* CPL *induced by the goal formula* $G$

# The calculus **FCL**($G$)

Example

Let $G_1$ be the goal formula $p \vee \neg p$.

- $\mathrm{Sf}(G_1) \;=\; \{\, G_1,\ \neg p,\ p \,\}$
- The calculus **FCL**($G_1$) consists of the following specialized rules

$$\frac{}{p \vdash p}\ \mathrm{Ax} \qquad \frac{\Gamma \vdash p, \Delta}{\Gamma \vdash p \vee \neg p, \Delta}\ R\vee \qquad \frac{\Gamma \vdash \neg p, \Delta}{\Gamma \vdash p \vee \neg p, \Delta}\ R\vee$$

$$\frac{\Gamma_1, p \vdash \Delta_1 \qquad \Gamma_2, \neg p \vdash \Delta_2}{\Gamma_1, \Gamma_2, p \vee \neg p \vdash \Delta_1, \Delta_2}\ L\vee \qquad \frac{\Gamma \vdash p, \Delta}{\Gamma, \neg p \vdash \Delta}\ L\neg \qquad \frac{\Gamma, p \vdash \Delta}{\Gamma \vdash \neg p, \Delta}\ R\neg$$

$\Gamma \subseteq \mathrm{Sf}(G_1),\ \Delta \subseteq \mathrm{Sf}(G_1)$

# The calculus **FCL**($G$)

- The derivation of $p \vee \neg p$ shown above is a derivation in the calculus **FCL**($G_1$)
- The non-terminating derivation

$$\begin{array}{lll} (1) & \mathrm{Ax} & p \vdash p \\ (2) & R\vee\ (1) & p \vdash p \vee p_1 \\ (3) & R\vee\ (2) & p \vdash p \vee p_1 \vee p_2 \\ (4) & R\vee\ (3) & p \vdash p \vee p_1 \vee p_2 \vee p_3 \\ & \vdots & \end{array}$$

cannot be build in **FCL**($G_1$). Indeed, none of the introduced formulas

$$p \vee p_1,\ p \vee p_1 \vee p_2,\ p \vee p_1 \vee p_2 \vee p_3,\ \cdots$$

is a subformula of $G_1 = p \vee \neg p$, hence none of the applications of rule $R\vee$ is allowed in **FCL**($G_1$).

# The calculus **FCL**($G$)

The calculus **FCL**($G$) enjoys the Finite Rule Property:

- $\surd$ **FCL**($G$) has a finite number of axioms.
- $\surd$ Given a finite number of sequents, there is only a finite number of rules of **FCL**($G$) applicable to them.

Only finitely many sequents can be proved in **FCL**($G$).
Thus, proof-search is terminating.

- Remark

  We can further restrict the proof-search space by exploiting the usual partition between left (positive) and right (negative) subformulas of $G$.
  For instance, the rule

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\, R\vee$$

  is allowed iff $A \vee B$ is a right subformula of $G$.

# Forward proof-search for **FCL**($G$)

A naive proof-search strategy for **FCL**($G$) can be implemented as follows.

We keep a *database* DB of proved sequents.

- Start

  Add to DB all the axioms of **FCL**($G$).

- Main Loop

  If DB contains sequents $\sigma_1, \ldots, \sigma_n$ and

  $$\frac{\sigma_1 \quad \cdots \quad \sigma_n}{\sigma}$$

  is a rule of **FCL**($G$), then add $\sigma$ to DB.

- Stop

  the sequent $\vdash G$ is get ($G$ is proved) OR
  DB is saturated (no new sequent can be added to DB).

Note that the procedure is terminating.

# Forward proof-search for **FCL**($G$)

Example (unprovable formula)

$G_2 = p \wedge q \qquad \mathrm{Sf}(G_2) = \{ p \wedge q,\ p,\ q \}$

| | | | |
|---|---|---|---|
| Iteration 1 | (1) | Ax | $p \vdash p$ |
| | (2) | Ax | $q \vdash q$ |
| Iteration 2 | (3) | $L\wedge$ (1) | $p \wedge q \vdash p$ |
| | (4) | $L\wedge$ (2) | $p \wedge q \vdash q$ |
| | (5) | $R\wedge$ (1) (2) | $p,\ q \vdash p \wedge q$ |
| Iteration 3 | (6) | $L\wedge$ (5) | $p \wedge q, q \vdash p \wedge q$ |
| | (7) | $L\wedge$ (5) | $p, p \wedge q \vdash p \wedge q$ |
| | (8) | $R\wedge$ (3) (4) | $p \wedge q \vdash p \wedge q$ |

The database is saturated.

For instance, by applying $L\wedge$ to (6), we again get (8).

$$\frac{p \wedge q, q \vdash p \wedge q \quad (6)}{p \wedge q, p \wedge q \vdash p \wedge q \quad (8)}\ L\wedge$$

# Forward proof-search for **FCL**($G$)

- The formula $G_2 = p \wedge q$ is not provable in **FCL**($G_2$) (the sequent $\vdash G_2$ does not belong to DB).
- Note that DB contains some redundancies.

  For instance, DB contains:

  $$(6)\ p \wedge q, q \vdash p \wedge q \qquad\qquad (8)\ p \wedge q \vdash p \wedge q$$

  However, (6) is *subsumed* by (8), in the sense that (8) implies (6) (while the converse does not hold). Indeed:

  if $\Gamma \vdash \Delta$ is valid and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \Delta$ is valid *(weakening)*

  We can rule out (6) from DB without losing completeness of proof-search.
- To improve proof-search efficiency, it is convenient to implement redundancies check, such as subsumption tests, so to avoid to keep redundant sequents in DB.

# Forward proof-search for **FCL**($G$)

$$G_2 = p \wedge q \qquad \mathrm{Sf}(G_2) = \{\, p \wedge q,\ p,\ q \,\}$$

When we add sequent (8), we can delete both (6) and (7) since both of them are subsumed by (8)

| | | | | |
|---|---|---|---|---|
| Iteration 1 | (1) | Ax | $p \vdash p$ | |
| | (2) | Ax | $q \vdash q$ | |
| Iteration 2 | (3) | $L\wedge$ (1) | $p \wedge q \vdash p$ | |
| | (4) | $L\wedge$ (2) | $p \wedge q \vdash q$ | |
| | (5) | $R\wedge$ (1) (2) | $p,\ q \vdash p \wedge q$ | |
| Iteration 3 | ~~(6)~~ | $L\wedge$ (5) | ~~$p \wedge q, q \vdash p \wedge q$~~ | subs. by (8) |
| | ~~(7)~~ | $L\wedge$ (5) | ~~$p, p \wedge q \vdash p \wedge q$~~ | subs. by (8) |
| | (8) | $R\wedge$ (3) (4) | $p \wedge q \vdash p \wedge q$ | |

# The Universal Recipe of Inverse Method

A. Degtyarev and A. Voronkov. *The inverse method*.
Handbook of Automated Reasoning, 2001.

- Goal

  Prove a formula $G$ *(goal formula)*.
- Calculus

  Design a specialized calculus $\mathbf{C}_G$ admitting terminating proof-search.
- Forward proof-search

  Forward apply the rules of $\mathbf{C}_G$ starting from axioms until possible (saturation process).

# Cooking it ...

- Classical and Intuitionistic Logic [Handbook AR, 2001]
- Logic of Bunched Implication [Donelly et al., LPAR 2004]
- Many-valued logics [Voronkov et al., MICAI 2013]
- A significant investigation about Intuitionistic Logic is presented in

  *K. Chaudhuri and F. Pfenning. A focusing inverse method theorem prover for first-order linear logic. CADE 2005*

  *K. Chaudhuri, F. Pfenning, and G. Price. A logical characterization of forward and backward chaining in the inverse method. IJCAR 2006.*

  Here focused calculi and polarization of formulas are exploited to reduce the search spaces in forward proof-search.

  These techniques are at the heart of the design of the prover Imogen

  *S. McLaughlin and F. Pfenning. Imogen: Focusing the polarized inverse method for intuitionistic propositional logic. LPAR 2008.*

# Outline

# Forward proof-search in refutation calculi

In all the mentioned papers, the inverse method has been exploited to prove the *validity* of a goal formula in a specific logic.

Here we follow the dual approach:

We propose forward calculi $\mathbf{C}_G$ to derive the non-validity of a goal formula $G$ in a logic $L$.

Thus, $\mathbf{C}_G$ is a forward refutation calculus for $L$.

- We focus on Intuitionistic Propositional Logic ($\mathrm{IPL}$) and we present a forward refutation calculus **FRJ**($G$) for $\mathrm{IPL}$.

  *C. Fiorentini and M. Ferrari. A Forward Unprovability Calculus for Intuitionistic Propositional Logic. TABLEAUX 2017, LNAI, vol. 10501, pp. 114-130, Springer, 2017.*

  *C. Fiorentini and M. Ferrari. Duality between unprovability and provability in forward proof-search for Intuitionistic Propositional Logic. arXiv:1804.06689, 2018.*

- We sketch some applications to modal logics.

# Forward refutation in IPL

We present a forward calculus **FRJ**($G$) to derive the non-validity of a goal formula $G$ in IPL.

$G$ is provable in **FRJ**($G$) $\iff$ $G \notin$ IPL

- If $G$ is provable in **FRJ**($G$):
  - $\surd$ from the derivation we extract a "small" Kripke countermodel for $G$, witnessing the non-validity of $G$ in IPL.
- If $G$ is not provable in **FRJ**($G$):
  - $\surd$ we get a saturated DB
  - $\surd$ by exploiting it, we build a derivation of $G$ in a standard sequent calculus for IPL, witnessing the validity of $G$ in IPL.

# Notation

- $\mathcal{V}$ is a set of propositional variables $p$, $q$, $p_1$, $p_2$, ...
- The language $\mathcal{L}$ based on $\mathcal{V}$ is the set of formulas $A$, $B$, ... such that:

$$\begin{array}{lll} A, B & ::= & \bot \mid p \mid A \wedge B \mid A \vee B \mid A \supset B \qquad p \in \mathcal{V} \\ \neg A & ::= & A \supset \bot \end{array}$$

- A Kripke model is a structure $\mathcal{K} = \langle P, \leq, \rho, V \rangle$, where:
  - $\langle P, \leq \rangle$ is a finite poset with minimum $\rho$ (root)
  - $V : P \to 2^{\mathcal{V}}$ is a function such that $\alpha \leq \beta$ implies $V(\alpha) \subseteq V(\beta)$
  - $\Vdash \subseteq P \times \mathcal{L}$ is the forcing relation:
    - $\alpha \nVdash \bot$
    - $\alpha \Vdash p$ iff $p \in V(\alpha)$
    - $\alpha \Vdash A \wedge B$ iff $\alpha \Vdash A$ and $\alpha \Vdash B$
    - $\alpha \Vdash A \vee B$ iff $\alpha \Vdash A$ or $\alpha \Vdash B$
    - $\alpha \Vdash A \supset B$ iff, for every $\beta \in P$ s.t. $\alpha \leq \beta$, $\beta \nVdash A$ or $\beta \Vdash B$

# Towards a Forward Refutation Calculus for $G$

- Sequents have the form

$$\Gamma \Rightarrow A \qquad\qquad \Gamma \cup \{A\} \subseteq \mathrm{Sf}(G)$$

- Soundness

  If $\Gamma \Rightarrow A$ is provable in **FRJ**$(G)$, then the sequent $\Gamma \Rightarrow \Delta$ is non-valid, namely:

  $\surd$ the formula $\bigwedge \Gamma \supset A$ is non-valid in $\mathrm{IPL}$

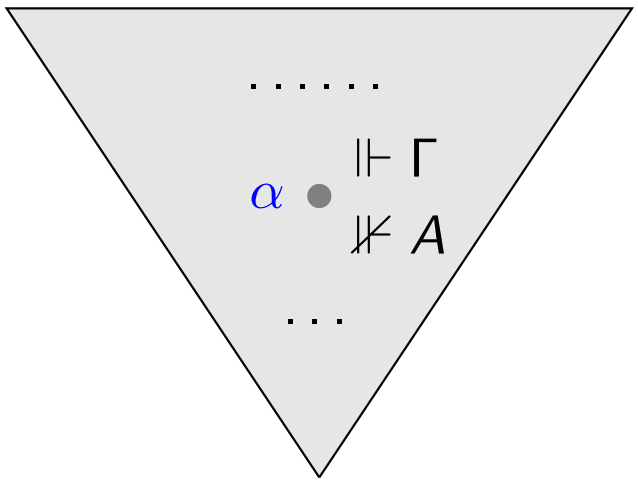
  This means that:

  $\surd$ the formula $A$ is not provable from formulas $\Gamma$ in $\mathrm{IPL}$

# Towards a Forward Refutation Calculus for $G$

- Soundness (semantic)

  $\surd$ if $\Gamma \Rightarrow A$ is provable in **FRJ**($G$), there exists a world $\alpha$ of a Kripke model such that:




All the formulas in $\Gamma$ are forced in $\alpha$

$A$ is not forced in $\alpha$

- Completeness

  If $G$ is non-valid in $\mathrm{IPL}$, then a sequent of the form

$$\Gamma \Rightarrow G$$

  is provable in **FRJ**($G$). Note that the set $\Gamma$ might be non-empty

# Towards a Forward Refutation Calculus for $G$

- Axioms

  In **FCL**($G$) axioms have the form

  $$p \vdash p \qquad\qquad p\text{: propositional variable}$$

  Since **FRJ**($G$) is a refutation calculus, axioms are unprovable sequents (in $\mathrm{IPL}$) only containing propositional variables and $\bot$:

  $$p_1, \ldots, p_n \Rightarrow q \qquad q \neq p_1, \ldots, q \neq p_n$$
  $$p_1, \ldots, p_n \Rightarrow \bot$$

  where $p_1, \ldots, p_n$, $q$ are propositional variables.

# Towards a Forward Refutation Calculus for $G$

Rules must preserve unprovability in $\mathrm{IPL}$

- Rule for $R\wedge$ (right and)

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \wedge B}\ R\wedge$$

If $A$ is not provable from $\Gamma$, then
$A \wedge B$ is not provable from $\Gamma$

- Rule for $L\vee$ (left or)

$$\frac{A, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\ L\vee$$

If $C$ is not provable from $\{A\} \cup \Gamma$, then
$C$ is not provable from $\{A \vee B\} \cup \Gamma$
*(Inversion Principle for left $\vee$)*

# Towards a Forward Refutation Calculus for $G$

## Tricky task

*How to cope with rules having more than one premise?*

- **FCL**($G$)

  Since rules must preserve provability in $\mathrm{CPL}$, side formulas must be gathered.

$$\frac{\Gamma_1 \vdash A, \Delta_1 \qquad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B,\ \Delta_1, \Delta_2}\ R\wedge$$

- Unprovability forward calculus

  Since rules must preserve unprovability in $\mathrm{IPL}$, side formulas must be intersected.

  Apparently, the rule $R\vee$ should be:

$$\frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \cap \Delta \Rightarrow A \vee B}\ R\vee$$

If $A$ is not provable from $\Gamma$ and $B$ is not provable from $\Delta$, then $A \vee B$ is not provable from $\Gamma \cap \Delta$

# Towards a Forward Refutation Calculus for $G$

The alleged rule for right or is unsound!

Trivial counterexample

$$\frac{\overbrace{p \vee q}^{\Gamma} \Rightarrow p \qquad \overbrace{p \vee q}^{\Delta} \Rightarrow q}{\underbrace{p \vee q}_{\Gamma \cap \Delta} \Rightarrow p \vee q}\, R\vee$$

- Premises
  $p$ is not provable from $p \vee q$
  $q$ is not provable from $p \vee q$
- Conclusion
  $p \vee q$ is provable from $p \vee q$

Thus, the rule does not preserve unprovability.

The problem is that intersection $\Gamma \cap \Delta$ is too big, we need a more clever strategy to join sequents.

This leads to the Forward Refutation calculus **FRJ**($G$).

# The calculus **FRJ**($G$)

- We introduce two kinds of sequent:
  - Regular sequents $\Gamma \Rightarrow C$
  - Irregular sequents $\Sigma\,;\,\Theta \rightarrow C$
- Formulas occurring in the sequents are subformulas of the goal formula $G$
- In the left, only atoms and implications.
- There are no left rules, but only rules to introduce the connectives $\wedge$, $\vee$, $\supset$ in the right and the multi-premise rules $\bowtie^{\mathrm{At}}$ and $\bowtie^{\vee}$ to join sequents.
- $G$ is *provable* in **FRJ**($G$) iff there exists an **FRJ**($G$)-derivation of a regular sequent of the form $\Gamma \Rightarrow G$.

**Theorem (Soundness and Completeness of FRJ($G$))**

*$G$ is provable in* **FRJ**($G$) $\iff$ *$G$ is not valid in* IPL

# The calculus **FRJ**($G$)

- Rule $\vee$

  This rule has two irregular sequents $\sigma_1$ and $\sigma_2$ as premises and yields an irregular sequent $\sigma$ introducing an $\vee$-formula in the right.

  $\Sigma$-*sets are preserved,* $\Theta$-*sets are intersected.*

$$\frac{\sigma_1 = \Sigma_1 ; \Theta_1 \to C_1 \qquad \sigma_2 = \Sigma_2 ; \Theta_2 \to C_2}{\sigma = \Sigma_1, \Sigma_2 ; \Theta_1 \cap \Theta_2 \to C_1 \vee C_2}\ \vee \qquad \begin{array}{l} \Sigma_1 \subseteq \Sigma_2 \cup \Theta_2 \\ \Sigma_2 \subseteq \Sigma_1 \cup \Theta_1 \end{array}$$

In the wrong $\vee$-rule:

$$\mathrm{Left}(\sigma) = \mathrm{Left}(\sigma_1) \cap \mathrm{Left}(\sigma_2)$$

Now:

$$\mathrm{Left}(\sigma) \subseteq \mathrm{Left}(\sigma_1) \cap \mathrm{Left}(\sigma_2)$$

# The calculus **FRJ**($G$)

- Join rules

  Join rules are multi-premise rules allowing the introduction on the right of an atomic formula (rule $\bowtie^{\mathrm{At}}$) or a disjunction (rule $\bowtie^{\vee}$).

- The Join rule $\bowtie^{\mathrm{At}}$

  It introduces a formula $F \in \mathcal{V} \cup \{\bot\}$ in the right.
  As in rule $\vee$, $\Sigma$-sets are gathered and $\Theta$-sets intersected.

$$\sigma_j \;=\; \underbrace{\Sigma_j^{\mathrm{At}}, \Sigma_j^{\supset}}_{\Sigma_j} ; \underbrace{\Theta_j^{\mathrm{At}}, \Theta_j^{\supset}}_{\Theta_j} \to A_j \qquad \text{where } \Sigma_j^{\mathrm{At}} \cup \Theta_j^{\mathrm{At}} \subseteq \mathcal{V} \text{ and } \Sigma_j^{\supset} \cup \Theta_j^{\supset} \subseteq \mathcal{L}^{\supset}$$

$$\frac{\sigma_1 \qquad \cdots \qquad \sigma_n}{\Sigma^{\mathrm{At}},\, \Theta^{\mathrm{At}} \setminus \{F\},\, \Sigma^{\supset},\, \Theta^{\supset} \Rightarrow F}\ \bowtie^{\mathrm{At}}$$

$\Sigma_i \subseteq \Sigma_j \cup \Theta_j$, for every $i \neq j$
$X \supset Y \in \Sigma^{\supset}$ implies $X \in \{A_1, \ldots, A_n\}$
$F \notin \Sigma^{\mathrm{At}}$

$$\begin{aligned}
\Sigma^{\mathrm{At}} &= \bigcup_{1 \leq j \leq n} \Sigma_j^{\mathrm{At}} \\
\Theta^{\mathrm{At}} &= \bigcap_{1 \leq j \leq n} \Theta_j^{\mathrm{At}} \\
\Sigma^{\supset} &= \bigcup_{1 \leq j \leq n} \Sigma_j^{\supset} \\
\Theta^{\supset} &= \{\, X \supset Y \in \bigcap_{1 \leq j \leq n} \Theta_j^{\supset} \mid X \in \{A_1, \ldots, A_n\} \,\}
\end{aligned}$$

# The calculus **FRJ**($G$)

$$\frac{}{\bar{\Gamma}^{\mathrm{At}} \setminus \{F\} \Rightarrow F}\ \mathrm{Ax}\Rightarrow \qquad \frac{}{\cdot\,;\, \bar{\Gamma}^{\mathrm{At}} \setminus \{F\}, \bar{\Gamma}^{\supset} \to F}\ \mathrm{Ax}\to \qquad F \in \mathcal{V} \cup \{\bot\}$$

$$\frac{\Gamma \Rightarrow A_k}{\Gamma \Rightarrow A_1 \wedge A_2}\ \wedge \qquad \frac{\Sigma\,;\,\Theta \to A_k}{\Sigma\,;\,\Theta \to A_1 \wedge A_2}\ \wedge \quad k \in \{1,2\}$$

$$\frac{\Sigma_1\,;\,\Theta_1 \to C_1 \qquad \Sigma_2\,;\,\Theta_2 \to C_2}{\Sigma_1, \Sigma_2\,;\,\Theta_1 \cap \Theta_2 \to C_1 \vee C_2}\ \vee \qquad \begin{array}{l}\Sigma_1 \subseteq \Sigma_2 \cup \Theta_2 \\ \Sigma_2 \subseteq \Sigma_1 \cup \Theta_1\end{array}$$

$$\frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\ \supset\in \quad A \in \mathcal{C}l(\Gamma) \qquad \frac{\Sigma\,;\,\Theta, \Lambda \to B}{\Sigma, \Lambda\,;\,\Theta \to A \supset B}\ \supset\in \quad \begin{array}{l}\Theta \cap \Lambda = \emptyset \\ A \in \mathcal{C}l(\Sigma \cup \Lambda)\end{array}$$

$$\frac{\Gamma \Rightarrow B}{\cdot\,;\,\Theta \to A \supset B}\ \supset\notin \qquad \begin{array}{l}\Theta \subseteq \mathcal{C}l(\Gamma) \cap \bar{\Gamma} \\ A \in \mathcal{C}l(\Gamma) \setminus \mathcal{C}l(\Theta)\end{array}$$

Let, for $1 \leq j \leq n$, $\sigma_j = \underbrace{\Sigma_j^{\mathrm{At}}, \Sigma_j^{\supset}}_{\Sigma_j}\,;\,\underbrace{\Theta_j^{\mathrm{At}}, \Theta_j^{\supset}}_{\Theta_j} \to A_j$ and $\Upsilon = \{A_1, \ldots, A_n\}$

$$\frac{\sigma_1 \quad \cdots \quad \sigma_n}{\Sigma^{\mathrm{At}}, \Theta^{\mathrm{At}} \setminus \{F\}, \Sigma^{\supset}, \Theta^{\supset} \Rightarrow F}\ \bowtie^{\mathrm{At}} \qquad \begin{array}{l}\Sigma_i \subseteq \Sigma_j \cup \Theta_j, \text{ for every } i \neq j \\ Y \supset Z \in \Sigma^{\supset} \text{ implies } Y \in \Upsilon\end{array}$$

$$\frac{\sigma_1 \quad \cdots \quad \sigma_n}{\Sigma^{\mathrm{At}}, \Theta^{\mathrm{At}}, \Sigma^{\supset}, \Theta^{\supset} \Rightarrow C_1 \vee C_2}\ \bowtie^{\vee} \qquad \begin{array}{l}\Sigma_i \subseteq \Sigma_j \cup \Theta_j, \text{ for every } i \neq j \\ Y \supset Z \in \Sigma^{\supset} \text{ implies } Y \in \Upsilon \\ \{C_1, C_2\} \subseteq \Upsilon\end{array}$$

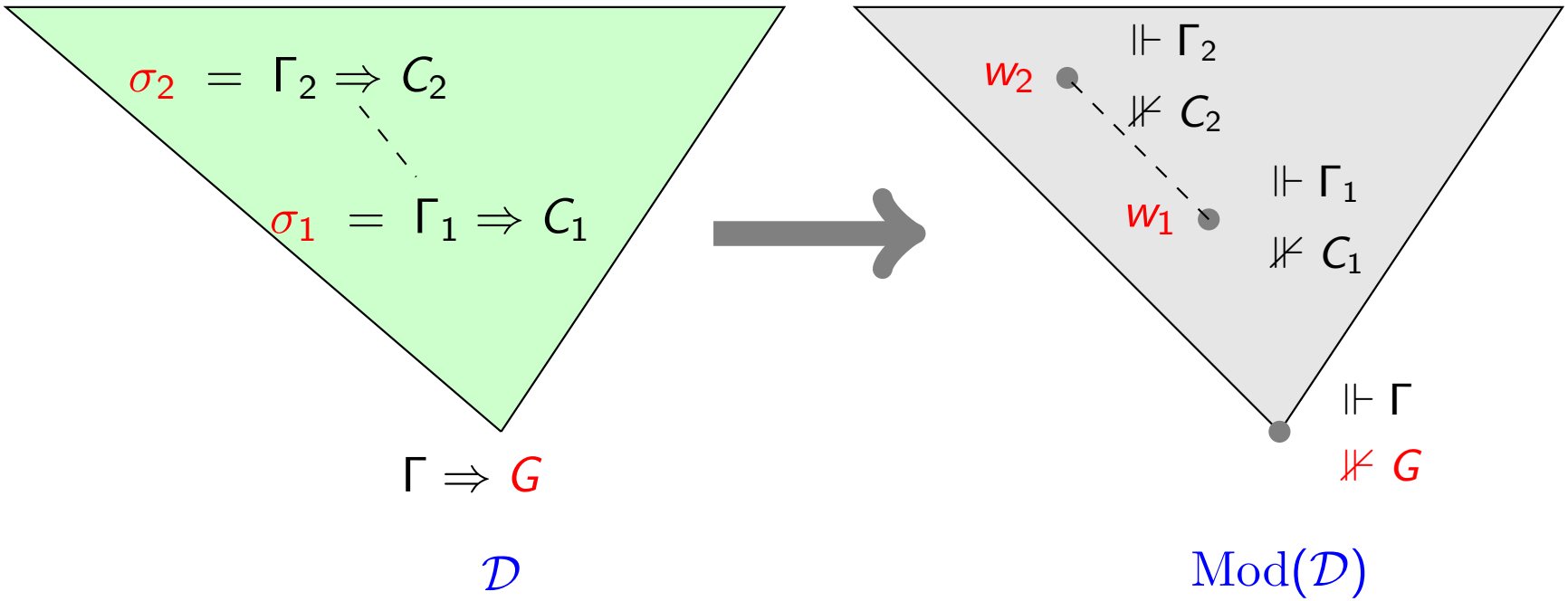
# The calculus **FRJ**($G$)

Let $G$ be provable in **FRJ**($G$).

- There exists an **FRJ**($G$)-derivation $\mathcal{D}$ of $\Gamma \Rightarrow G$
- From $\mathcal{D}$ we extract a Kripke model $\mathrm{Mod}(\mathcal{D})$ closely related to $\mathcal{D}$.

  At the root of $\mathrm{Mod}(\mathcal{D})$ all the formulas in $\Gamma$ are forced, whereas $G$ is not forced.

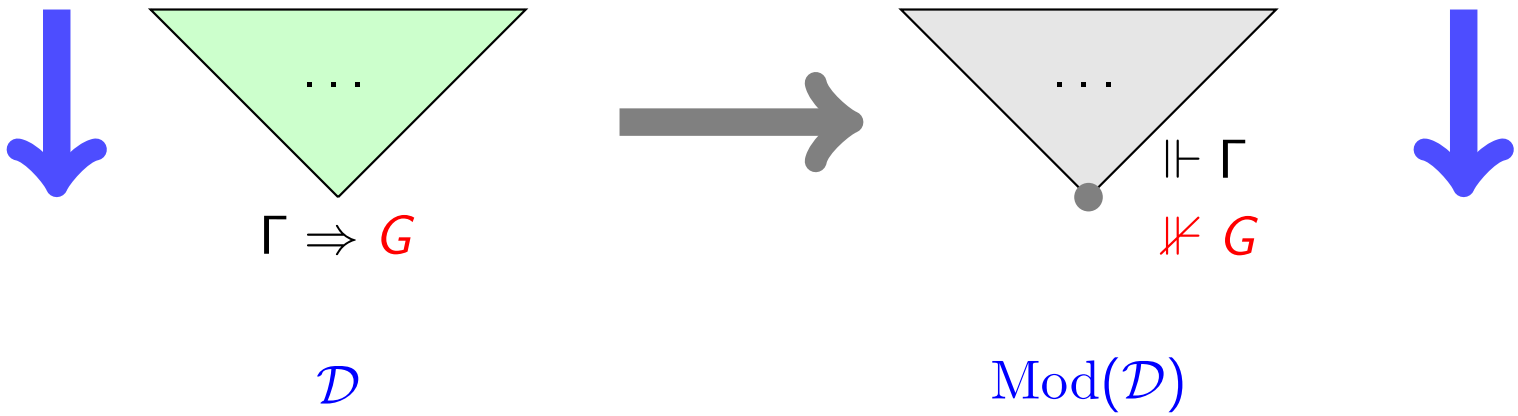
Accordingly, $\mathrm{Mod}(\mathcal{D})$ is a countermodel for $G$.



# The calculus **FRJ**($G$)

In forward-proof search, $\mathcal{D}$ is built top-down, starting from axioms.

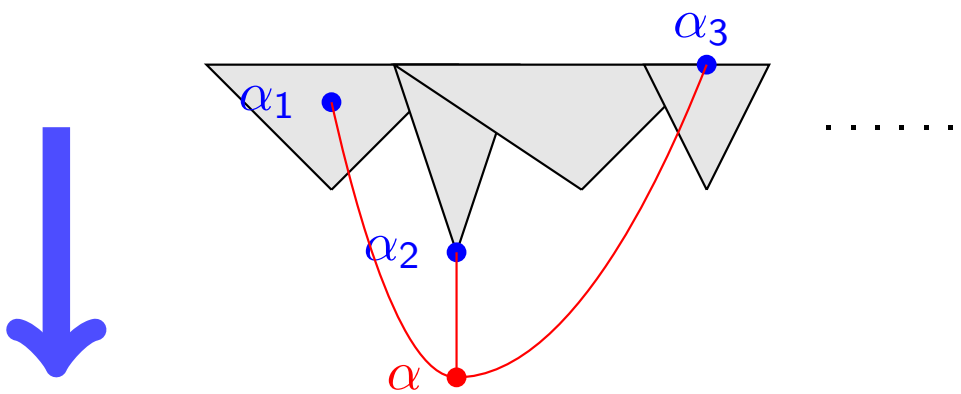
This corresponds to a top-down construction strategy of the countermodel $\mathrm{Mod}(\mathcal{D})$, starting from the top-worlds towards the root.



# The calculus **FRJ**($G$)

Join rules correspond to a step in *downward* countermodel construction:

- ⋆ we select $n \geq 1$ worlds $\alpha_1, \ldots, \alpha_n$ and we add a new world $\alpha$ having as immediate successors the chosen worlds.



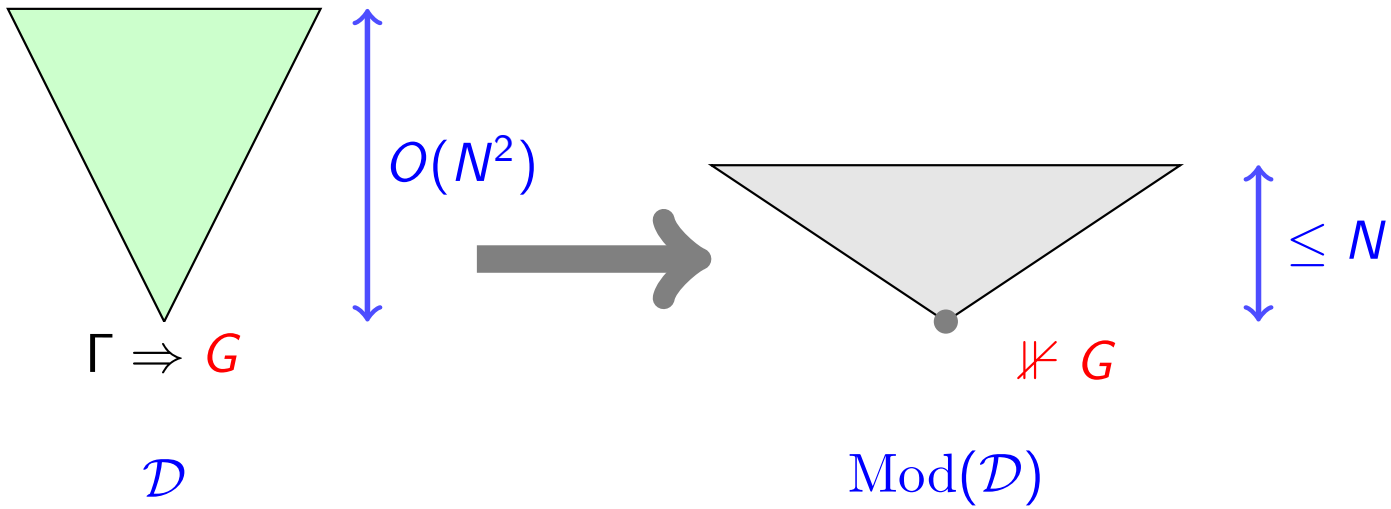

$\alpha$: new world having the chosen worlds $\alpha_1$, $\alpha_2$, $\alpha_3$ as immediate successors.

# The calculus **FRJ**($G$)

Let $\mathcal{D}$ be an **FRJ**($G$)-derivation of $G$ and
$N$ the size of $G$ (= number of symbols occurring in $G$).

Then:

- $\text{height}(\mathcal{D}) \;=\; O(\,N^2\,)$
- $\text{height}(\,\text{Mod}(\mathcal{D})\,) \;\leq\; N$



# Proof-search

The naive proof-search procedure is not efficient:

- Join rules must be applied to every combination of $n \geq 1$ sequents.
- Too many redundant sequents are generated.

To reduce redundancies:

- ⋆ We introduce a *subsumption* relation between sequents.
- ⋆ We tweak the proof-search procedure so that `DB` never contains pairs of sequents subsuming each other *(subsumption check)*.

  Indeed, if both $\sigma_1$ and $\sigma_2$ belong to `DB` and $\sigma_1$ subsumes $\sigma_2$, then $\sigma_2$ is redundant and can be safely removed.

We have implemented `frj`, a Java prototype of our proof-search procedure based on JTabWb (a Java framework for developing provers)

`http://github.com/ferram/jtabwb_provers/`

# Countermodels

Our proof/countermodel-search procedure is dual to the standard bottom-up methods, which mimic the backward application of rules.

This different approach has a significant impact on the outcome:

- Backward procedures

  Countermodels are always trees, which might contain many redundancies (the same sequent might occur many times in the tree).

- Forward procedures

  Prone to re-use sequents as much as possible and to not generate redundant ones (the DB does not contain duplications)
  Thus the obtained countermodels are in general very concise.

# Example: Anti-Scott principle

$$G \;=\; (\,((\neg\neg p \supset p) \supset (\neg p \vee p)) \;\supset\; (\neg\neg p \vee \neg p)\,) \;\;\supset\;\; ((\neg\neg p \supset p) \vee \neg\neg p)$$

$$G \;=\; S \;\supset\; ((\neg\neg p \supset p) \vee \neg\neg p)$$

$$S \;=\; H \supset (\neg\neg p \vee \neg p) \qquad H \;=\; (\neg\neg p \supset p) \supset (\neg p \vee p)$$

The goal $G$ is an instance of Anti-Scott principle (not valid in $\mathrm{IPL}$).

To prove the goal, `frj` runs 10 iterations of the main loop.

Legenda

| | |
|---|---|
| $sub(n)$: | sequent subsumed by sequent $n$ (backward subsumption) |
| $(n)$: | sequent needed to prove the goal |
| $(n)$: | sequent corresponding to a world of the countermodel |

- Iteration 0 (axioms)

| | | | |
|---|---|---|---|
| $sub(15)$ | ~~$(0)$~~ | $\mathrm{Ax}_{\Rightarrow}$ | ~~$p \Rightarrow \bot$~~ |
| $sub(10)$ | ~~$(1)$~~ | $\mathrm{Ax}_{\Rightarrow}$ | ~~$\cdot \Rightarrow p$~~ |
| | $(2)$ | $\mathrm{Ax}_{\rightarrow}$ | $\cdot\,;\, p, \neg p, \neg\neg p, \neg\neg p \supset p, S \rightarrow \bot$ |
| | $(3)$ | $\mathrm{Ax}_{\rightarrow}$ | $\cdot\,;\, \neg p, \neg\neg p, \neg\neg p \supset p, S \rightarrow p$ |

# Example: Anti-Scott principle

- Iteration 1

| | | | |
|---|---|---|---|
| $sub(19)$ | ~~$(4)$~~ | $\supset_{\in}$ $(0)$ | ~~$p \Rightarrow \neg p$~~ |
| $sub(20)$ | ~~$(5)$~~ | $\supset_{\notin}$ $(0)$ | ~~$\cdot\,;\ \neg\neg p \supset p \to \neg p$~~ |
| | $(6)$ | $\supset_{\in}$ $(2)$ | $p\,;\ \neg p, \neg\neg p, \neg\neg p \supset p, S \to \neg p$ |
| | $(7)$ | $\supset_{\in}$ $(2)$ | $\neg p\,;\ p, \neg\neg p, \neg\neg p \supset p, S \to \neg\neg p$ |
| | $(8)$ | $\supset_{\in}$ $(3)$ | $\neg\neg p\,;\ \neg p, \neg\neg p \supset p, S \to \neg\neg p \supset p$ |
| $sub(17)$ | ~~$(9)$~~ | $\bowtie^{\mathrm{At}}$ $(3)$ | ~~$\neg p \Rightarrow \bot$~~ |
| $sub(18)$ | ~~$(10)$~~ | $\bowtie^{\mathrm{At}}$ $(3)$ | ~~$\neg p \Rightarrow p$~~ |

- Iteration 2

| | | | |
|---|---|---|---|
| $sub(24)$ | ~~$(11)$~~ | $\vee(5)(3)$ | ~~$\cdot\,;\ \neg\neg p \supset p \Rightarrow \neg p \vee p$~~ |
| | $(12)$ | $\vee(8)(7)$ | $\neg p, \neg\neg p\,;\ \neg\neg p \supset p, S \to (\neg\neg p \supset p) \vee \neg\neg p$ |
| $sub(21)$ | ~~$(13)$~~ | $\supset_{\in}$ $(9)$ | ~~$\neg p \Rightarrow \neg\neg p$~~ |
| $sub(22)$ | ~~$(14)$~~ | $\supset_{\notin}$ $(9)$ | ~~$\cdot\,;\ S \to \neg\neg p$~~ |
| | $(15)$ | $\bowtie^{\mathrm{At}}$ $(6)$ | $p, \neg\neg p \Rightarrow \bot$ |
| $sub(26)$ | ~~$(16)$~~ | $\bowtie^{\vee}$ $(3)(5)$ | ~~$\cdot \Rightarrow \neg p \vee p$~~ |
| | $(17)$ | $\bowtie^{\mathrm{At}}$ $(3)(7)$ | $\neg p, \neg\neg p \supset p \Rightarrow \bot$ |
| | $(18)$ | $\bowtie^{\mathrm{At}}$ $(3)(7)$ | $\neg p, \neg\neg p \supset p \Rightarrow p$ |

# Example: Anti-Scott principle

- Iteration 3

| | | | |
|---|---|---|---|
| | (19) | $\supset_{\in}$ (15) | $p, \neg\neg p \Rightarrow \neg p$ |
| | (20) | $\supset_{\notin}$ (15) | $\cdot\,;\, \neg\neg p, \neg\neg p \supset p, S \to \neg p$ |
| | (21) | $\supset_{\in}$ (17) | $\neg p, \neg\neg p \supset p \Rightarrow \neg\neg p$ |
| | (22) | $\supset_{\notin}$ (17) | $\cdot\,;\, \neg\neg p \supset p, S \to \neg\neg p$ |
| *sub*(32) | ~~(23)~~ | $\supset_{\in}$ (11) | ~~$\neg\neg p \supset p\,;\, \cdot \to H$~~ |

- Iteration 4

| | | | |
|---|---|---|---|
| | (24) | $\vee$(20)(3) | $\cdot\,;\, \neg\neg p, \neg\neg p \supset p, S \to \neg p \vee p$ |
| | (25) | $\bowtie^{\mathrm{At}}$ (20) | $\neg\neg p \Rightarrow p$ |
| | (26) | $\bowtie^{\vee}$ (3)(20) | $\neg\neg p \Rightarrow \neg p \vee p$ |
| *sub*(37) | ~~(27)~~ | $\bowtie^{\vee}$ (3)(20)(22) | ~~$\neg\neg p \supset p \Rightarrow \neg p \vee p$~~ |

- Iteration 5

| | | | |
|---|---|---|---|
| | (28) | $\supset_{\in}$ (25) | $\neg\neg p \Rightarrow \neg\neg p \supset p$ |
| | (29) | $\supset_{\notin}$ (25) | $\cdot\,;\, S \to \neg\neg p \supset p$ |
| *sub*(38) | ~~(30)~~ | $\supset_{\in}$ (27) | ~~$\neg\neg p \supset p \Rightarrow H$~~ |
| *sub*(39) | ~~(31)~~ | $\supset_{\notin}$ (27) | ~~$\cdot\,;\, \cdot \to H$~~ |
| | (32) | $\supset_{\in}$ (24) | $\neg\neg p \supset p\,;\, \neg\neg p, S \to H$ |

# Example: Anti-Scott principle

- Iteration 6

| | | | |
|---|---|---|---|
| | (33) | $\vee(29)(22)$ | $\cdot\,;\, S \to (\neg\neg p \supset p) \vee \neg\neg p$ |
| $sub(40)$ | ~~(34)~~ | $\bowtie^{\vee}\ (22)(29)$ | ~~$\cdot \Rightarrow (\neg\neg p \supset p) \vee \neg\neg p$~~ |
| | (35) | $\bowtie^{\mathrm{At}}\ (22)(32)$ | $\neg\neg p \supset p, S \Rightarrow \bot$ |
| | (36) | $\bowtie^{\mathrm{At}}\ (22)(32)$ | $\neg\neg p \supset p, S \Rightarrow p$ |
| | (37) | $\bowtie^{\vee}\ (3)(20)(22)(32)$ | $\neg\neg p \supset p, S \Rightarrow \neg p \vee p$ |

- Iteration 7

| | | |
|---|---|---|
| (38) | $\supset_{\in}\ (37)$ | $\neg\neg p \supset p, S \Rightarrow H$ |
| (39) | $\supset_{\notin}\ (37)$ | $\cdot\,;\, S \to H$ |

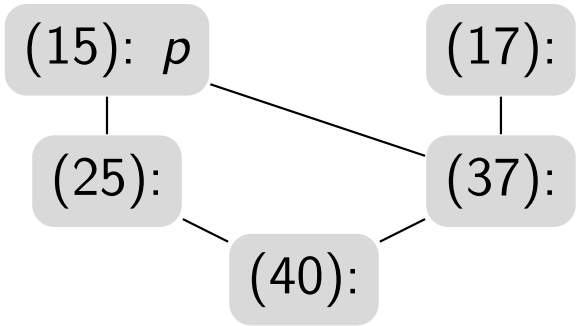
- Iteration 8

| | | |
|---|---|---|
| (40) | $\bowtie^{\vee}\ (22)(29)(39)$ | $S \Rightarrow (\neg\neg p \supset p) \vee \neg\neg p$ |

- Iteration 9 (Goal)

| | | |
|---|---|---|
| (41) | $\supset_{\in}\ (40)$ | $S \Rightarrow G$ |

# Example: Anti-Scott principle




(15) $p, \neg\neg p \Rightarrow \bot$

(17) $\neg p, \neg\neg p \supset p \Rightarrow \bot$

(25) $\neg\neg p \Rightarrow p$

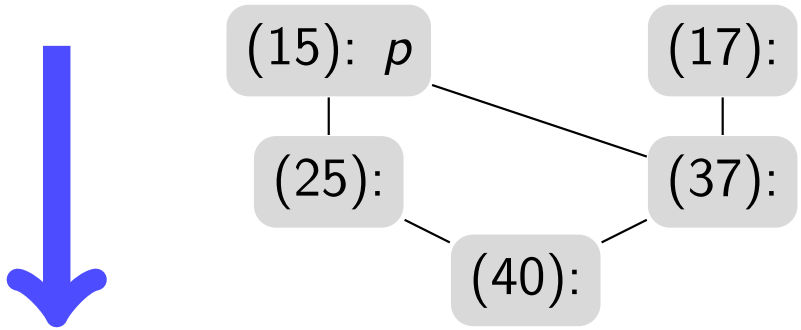
(37) $\neg\neg p \supset p \Rightarrow \neg p \vee p$

(40) $S \Rightarrow (\neg\neg p \supset p) \vee \neg\neg p$

$G = S \supset ((\neg\neg p \supset p) \vee \neg\neg p)$ $S = H \supset (\neg\neg p \vee \neg p)$ $H = (\neg\neg p \supset p) \supset (\neg p \vee p)$

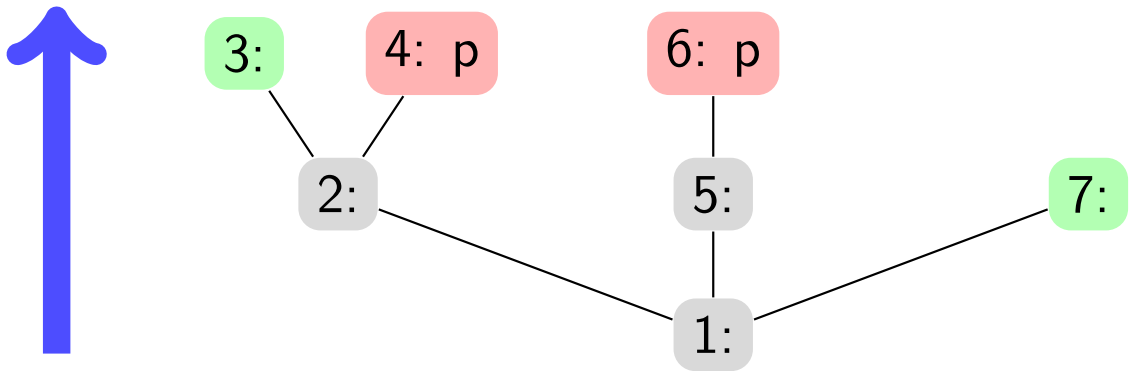
- At the end of the computation DB contains 38 sequents:
  - √ 15 sequents have been deleted by (backward) subsumption
  - √ 16 sequents are needed to prove the goal

# Example: Anti-Scott principle




The obtained model is minimal in the number of worlds and is *not a tree*, hence it cannot be obtained by standard bottom-up methods.

For instance, using `lsj`, a prover based on the calculus presented in [Ferrari et. al., JAR 2013] we get the following tree-shaped countermodel, which has *minimal height*, but contains some redundancies.




6 is a replica of 4

7 is a replica of 3

# Example: Nishimura formulas

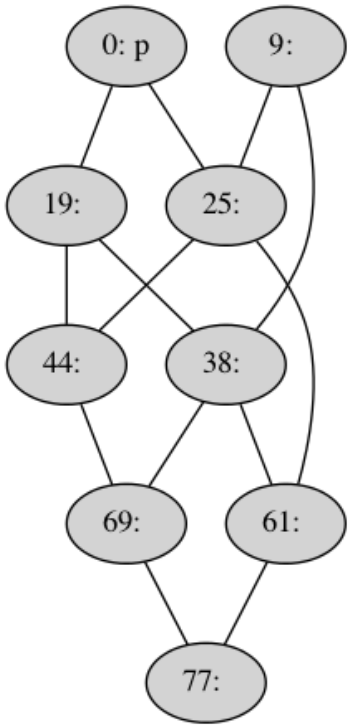
We get very concise models with one-variable Nishimura formulas:

$$N_1 = p \qquad N_{2n+3} = N_{2n+1} \vee N_{2n+2}$$
$$N_2 = \neg p \qquad N_{2n+4} = N_{2n+3} \supset N_{2n+1}$$

$N_9$ : equivalent to Anti-Scott principle

Indeed, `frj` yields the standard "tower-like" minimum countermodels.

Countermodel for $N_{17}$

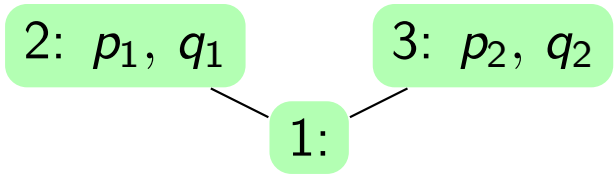


# On countermodels

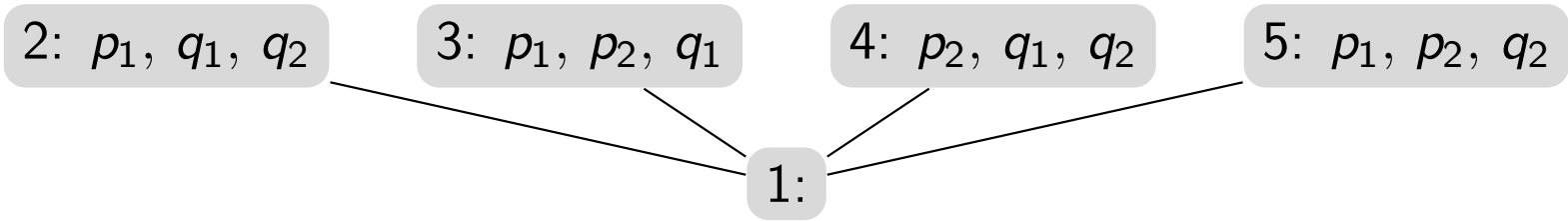
- We can tweak the proof-search strategy so to get countermodels having minimal height
- However, the countermodels might not be minimal. For instance:

$$G \;=\; (p_1 \supset p_2) \vee (p_2 \supset p_1) \vee (q_1 \supset q_2) \vee (q_2 \supset q_1)$$

Minimal Countermodel:




Countermodel $\mathcal{K}$ generated by `frj`:




- $\mathcal{K}$ has the same height of the minimal countermodel
- Final worlds of $\mathcal{K}$ have "maximal" forcing (only one prop. var. is not forced), thus we cannot simulate the minimal countermodel

# On saturated database

Whenever proof-search in **FRJ**($G$) fails, we get a *saturated database* DB for $G$, namely:

- If a sequent $\sigma$ is provable in **FRJ**($G$), there exists $\sigma'$ in DB such that $\sigma'$ subsumes $\sigma$.

We exploit DB to build a sequent derivation of $G$, so to constructively ascertain the validity of $G$.

To this aim, we introduce the sequent calculus **Gbu**($G$), a "focused" variant of the well-known sequent calculus **G3i**.

- $\surd$ **Gbu**($G$) can be viewed as the dual calculus of **FRJ**($G$)
- $\surd$ **Gbu**($G$) is closely related with the calculus presented in

  *M. Ferrari, C. Fiorentini, and G. Fiorino. A terminating evaluation-driven variant of G3i. TABLEAUX 2013.*

# On saturated database

- **G3i**

$$\frac{}{\Gamma, p \vdash p}\ \mathrm{Ax}_1 \qquad \frac{}{\bot, \Gamma \vdash C}\ \mathrm{Ax}_2$$

$$\frac{A, B, \Gamma \vdash C}{A \wedge B, \Gamma \vdash C}\ L\wedge \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ R\wedge$$

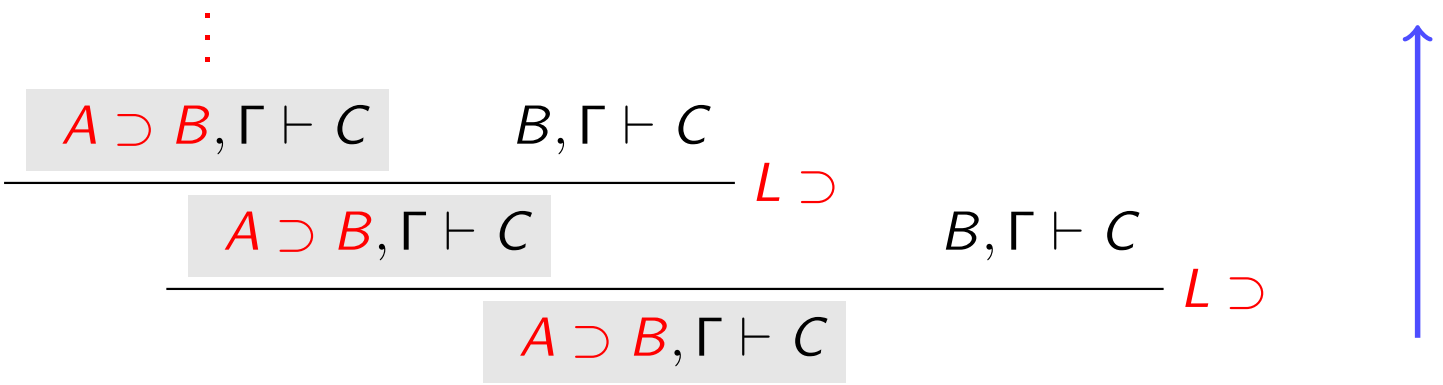
$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{A \vee B, \Gamma \vdash C}\ L\vee \qquad \frac{\Gamma \vdash A_k}{\Gamma \vdash A_1 \vee A_2}\ R\vee \qquad k = 0, 1$$

$$\frac{A \supset B, \Gamma \vdash A \qquad B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C}\ L\supset \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B}\ R\supset$$

- **Gbu**($G$) = **G3i** + labelled sequents (two kinds of sequents)
  + side conditions on some rule applications

# On saturated database

- In **G3i**, bottom-up proof search is not terminating.
  Indeed, **G3i** allows for unbounded applications of rule $L\supset$ of this kind:



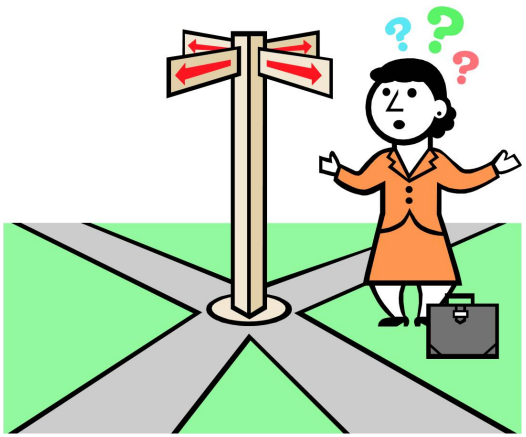

- In **Gbu**$(G)$ the number of applications of rule $L\supset$ is bounded by the size of the root sequent.

  Hence, bottom-up proof-search in **Gbu**$(G)$ is terminating

# On saturated database

In **Gbu**($G$) bottom-up proof-search in general requires backtracking:

$$\frac{\dots}{A_1 \supset B_1 , \dots , A_n \supset B_n \vdash p}\ L \supset ??$$



- We have to choose the main formula $A_j \supset B_j$ of $L \supset$ application.
- If we take the wrong way, we have to backtrack and try another choice.

# On saturated database

Example

$$\frac{\dots}{p_1,\ p_1 \supset p_2,\ p_3 \supset p_4 \vdash p_2}\ L\supset ??$$

We can choose $p_1 \supset p_2$ or $p_3 \supset p_4$.

- If we choose $p_3 \supset p_4$, proof search fails since the left-most premise is not provable:

$$\frac{\text{UNPROVABLE} \qquad\qquad p_1,\ p_1 \supset p_2,\ p_4 \vdash p_2}{p_1,\ p_1 \supset p_2,\ p_3 \supset p_4 \vdash p_2}\ L\supset$$

- To build a derivation, we have to backtrack and try the other way

$$\frac{\dfrac{}{p_1,\ p_1 \supset p_2,\ p_3 \supset p_4 \vdash p_1}\ \mathrm{Ax} \qquad \dfrac{}{p_1,\ p_2,\ p_3 \supset p_4 \vdash p_2}\ \mathrm{Ax}}{p_1,\ p_1 \supset p_2,\ p_3 \supset p_4 \vdash p_2}\ L\supset$$

# On saturated database

However, we can exploit the DB obtained at the end of proof-search to avoid backtracking and choose the right path.

To sum up:

- If $G$ is valid in $\mathrm{IPL}$, forward proof-search in **FRJ**($G$) fails.
- At the end of proof-search we obtain a saturated database DB.
- We can exploit DB to deterministically construct a sequent derivation of $G$ in **Gbu**($G$):

  *whenever a backtrack point occurs, ask* DB *the right way.*

Thus a saturated DB can be viewed as a proof-certificate of the validity of $G$.

# On saturated database

A dual remark has been issued in

> *S. McLaughlin and F. Pfenning. Imogen: Focusing the polarized inverse method for intuitionistic propositional logic. LPAR 2008.*

The authors introduce a forward (focused) sequent calculus for $\mathrm{IPL}$.

If proof-search for a goal $G$ fails, one gets a saturated database `DB`.

The authors claim that such a saturated `DB`

> *"may be considered a kind of countermodel for the goal sequent".*

But so far this issue has not been investigated.

# Recap on **FRJ**($G$)

- **FRJ**($G$) is a forward calculus to derive the unprovability of a goal formula $G$ in $\mathrm{IPL}$:
  - $\surd$ If $G$ is provable in **FRJ**($G$), from the derivation we can immediately extract a countermodel for $G$;
  - $\surd$ otherwise, we get a saturated `DB` which can be exploited to get a sequent-style derivation of $G$ in $\mathrm{IPL}$.

    Thus a saturated `DB` can be viewed as a proof-certificate of the validity of $G$ in $\mathrm{IPL}$.
- Advantages of forward vs. backward reasoning:
  - $\surd$ derivations are more concise since sequents are reused and not duplicated (subsumption tests)
  - $\surd$ countermodels are in general compact and have minimal height