Heinrich Wansing
Ruhr-University Bochum
Heinrich.Wansing@rub.de

Poznań Reasoning Week
Refutation Symposium
Adam Mickiewicz University
Poznań, 14-15 September 2018

Outline of the talk:

- Some general remarks about refutation
  - weak and strong refutability
  - refutation of formulas versus refutation of entailment statements
- A semantics and a natural deduction system for refutation as falsification
- Mention a recent result by Sergey Drobyshevich

Some general remarks about refutation

Tomasz Skura

Definition

Skura (2009) A refutation rule for a logic $\mathcal{L}$ is a rule R with the property that the complement of $\mathcal{L}$ (that is, the set $FOR - \mathcal{L}$) is R-closed, that is, for every inference $\mathcal{X}/A \in$ R we have: If $\mathcal{X} \subseteq FOR - \mathcal{L}$, then $A \in FOR - \mathcal{L}$.

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(Skura 2016) A refutation system is an axiomatic system for non-valid formulas. It consists of refutation axioms, which are some non-valid formulas, and refutation rules, which are some rules preserving non-validity.

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What is refutability?

Refutability as a property of formulas

classical logic

formulas that are not valid	formulas that are unsatisfiable
formulas that are not true (have a non-designted value) in some model	formulas that are not true in any model
formulas that are false in some model	formulas that are false in every model
falsifiable/refutable	strongly falsif./strongly refutable
strongly refutable implies refutable	
formulas whose negation is true in some model	formulas whose negation is true in every model
formulas whose negation is not false in some model	formulas whose negation is not false in any model

Consider **separated many-valued logics** in which a distinction is drawn between a set of *designated* (algebraic) values and a set of *anti-designated* (algebraic) values [Shramko and Wansing 2011], where it is not required that these sets are exhaustive or disjoint.

Then the followign situation may arise:

forrmulas not true (not designated) in at least one model	formulas not true (not designated) in any model	formulas false (anti-designated) in at least one model	formulas false (anti-designated) in every model
very weakly refutable	weakly refutable	refutable	strongly refutable

strongly refutable implies refutable
weakly refutable implies very weakly refutable

refutable does not imply very weakly refutable if formulas can be both desinated and anti-designated

Are there lany interesting logics containing formulas that are strongly refutable and true in at least one model?

There are *nontrivial* 'dialectical' logics, like the connexive logics C [Wansing 2005] or 2C [Wansing 2016] containing formulas that are both strongly refutable in the sense that in their relational (Kripke) semantis every state of every model supports their falsity and are also *valid* (i.e., every state of every model supports their truth).

There are **non-separated many-valued logics** in which the classially equivalent characterizations of the refutable formulas fail to be equivalent.

Consider the so-called ”Logic of paradox,” **LP.**

valid formulas	formulas that are not valid
receive a designated value in every model	formulas that are not true in some model (have a non-designted value in some model)
not refutable	refutable
~~formulas whose negation is false in every model~~	~~formulas whose negation is true (designated) in some model~~
$(p \vee {\sim} p)$ is valid ${\sim}\ (p \vee {\sim} p)$ has a designated value in some model	every formula can receive the *infectious value* and hence also every negated formula, but there *are* formulas that are not refutable in the sense of receiving a designated value in every model

In **LP** there are no formulas that receive an anti-designated value in every model (in this case also no formulas that receive a non-designated value in every model), that is, there is no formula that is strongly refutable (or weakly refutable, since both notions coincide). But perhaps the focus lies on formulas that are not true (not-designated) in at least one model anyway.

In any case in **LP** every formula is not anti-designated in some model (in this case every formula is not non-designated in some model). The notion of being not anti-designatable (or being not non-designatable) is *trivial*.

Given a not uncommom preoccupation with positive notions in comaprison to negative ones, this might seem unproblematic.

Nevertheless already in **LP** we are primarily interested not in the set of all valid formulas (i.e., the classical tautologies) or its complement but in the relation of valid consequence.

This interest in the relation of valid consequence is maybe more obvious in the case of Keene's strong three-valued logic, **K3**.

In **K3** there are no valid formulas, thus every formula is refutable (does not receive a designated value in any model). In **K3**, the notion of a refutable formula is *trivial*.

K3 provides one more reason to focus not on the notion of refutable formulas but instead on the notion of refutable entailment statements.

Refutabilit as a property of entailent statements $\Delta \models A$

classical logic

valid entailment perserves truth from the premises to the conclusion:

for every model *M*, if all premises are true in *M*, then so is the conclusion

and non-truth from the conclusion to the premises:

for every model *M*, if the conclusion is not true in *M*, then at least one of the premises is not true in *M*

and if non-truth means falsity, then for every model *M*, if the conclusion is false in *M*, then at least one of the premises is false in *M*

Thus, if falsity means non-truth, then in valid entailment falsity is preserved from the conclusion to at elast one of the premises.

If negation expresses non-truth, then valid entailment guarantees contraposition:

$$A \models B \text{ iff } {\sim} B \models {\sim} A$$

However, if negation expreses falsity and falsity does not amount to non-truth, negation is not guaranteed to validate contraposition.

In a four-valued setting, we may draw a number of distinctions:

valid entailment as preservation of support of truth

For every model *M*, if the truth of all premises is suported in *M*, then the truth of the conclusion is supported in *M*.

weak refutabilty

It is not the case that for every model *M*, if the truth of all premises is suported in *M*, then the truth of the conclusion is supported in *M*.

(There is a model *M* such that the truth of no premise is not supported in *M* but the truth of the conclusion is not supported in *M*.)

strong refutability

For every model *M*, if the falsity of all premises is suported in *M*, then the falsity of the conclusion is supported in *M*.

weak support of truth

It is not the case that for every model *M*, if the falsity of all premises is suported in *M*, then the falsity of the conclusion is supported in *M*.

(There is a model *M* such that the falsity of no premise is not supported in *M* but the falsity of the conclusion is not supported in *M*.)

When it comes to inferences, we would want to capture valid entailment by derivabilty and strong refutatbility by another relation, which we might want ot call dual derivabilty.

If we think of *constructive* derivabilty, we may consider intuitionistic logic, where derivability is internalized by intuitionistic implication.

We then have to supplement intuitionsitic derivability by it's dual and by a connective that internalizes dual derivabilty: constructive co-implication.

Thus, in a four-valued setting we may still be interested in the complement of the relation of valid entailment and the complement of the set of valid formulas., i.e., in weak refutability, but strong refutability is a genuine notion of refutability as well.

Semantics and natural deduction for refutation as falsification

Co-implication in Heyting-Brouwer logic

The language $\mathcal{L}_{\mathsf{BiInt}}$ of Heyting-Brouwer logic, also called "bi-intuitionistic logic", BiInt, extends the language of intuitionistic logic, Int, by a primitive binary co-implication connective $\prec$ and is defined in Backus–Naur form as follows:

$$A ::= \quad p \mid \bot \mid \top \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid (A \prec A).$$

where p is a propositional variable from some fixed denumerably infinite set Φ of sentential variables (atomic formulas).

The language $\mathcal{L}_{\mathsf{DualInt}}$ of dual intuitionistic propositional logic, DualInt, is $\mathcal{L}_{\mathsf{BiInt}}$ restricted to the connectives $\top$, $\bot$, $\wedge$, $\vee$, and $\prec$.

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In the relational semantics for BiInt, a state x from a Kripke model $\mathcal{M} = \langle I, \leq, v \rangle$ supports the truth of a co-implication $(A \prec B)$ ("B co-implies A") iff there is an "earlier" state x' such that x' supports the truth of A but fails to support the truth of B:

$\mathcal{M}, x \models (A \prec B)$ iff there exists $x' \leq x$ with $\mathcal{M}, x' \models A$ and $\mathcal{M}, x' \not\models B$.

The support of truth clause for implication is the intuitionistic one:

$$\mathcal{M}, x \models (A \to B) \text{ iff for every } x' \geq x : \mathcal{M}, x' \not\models A \text{ or } \mathcal{M}, x' \models B.$$

Every (no) state supports the truth of the truth (falsity) constant $\top$ ($\bot$).

A formula A is valid in a model for BiInt $\mathcal{M} = \langle I, \leq, v \rangle$ iff for every $x \in I$, $\mathcal{M}, x \models A$, and A is valid in BiInt iff A is valid in every model for BiInt.

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We write $\models_{\mathsf{BiInt}} A$ to mean that A is valid in BiInt. Let $\Delta \cup \{A\}$ be a set of formulas. Δ entails A in BiInt ($\Delta \models_{\mathsf{BiInt}} A$) iff for every model for BiInt $\mathcal{M} = \langle I, \leq, v \rangle$ and every $x \in I$, it holds that if the truth of every element of Δ is supported by x, then the truth of A is supported by x.

Now we may note that

$$A \models_{\mathsf{BiInt}} B \text{ iff } \top \models_{\mathsf{BiInt}} A \to B \text{ iff } A \prec B \models_{\mathsf{BiInt}} \bot.$$

It is in this sense that co-implication in BiInt internalizes preservation of non-truth from the conclusion of a valid inference (understood as logical consequence) to the premises.

Moreover, in the following sense co-implication in BiInt is the residual of disjunction with respect to entailment: $A \models_{\mathsf{BiInt}} B \vee C$ iff $A \prec B \models_{\mathsf{BiInt}} C$ iff $A \prec C \models_{\mathsf{BiInt}} B$.

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Co-implication in 2Int

The bi-intuitionistic logic 2Int introduced in (Wansing 2013, Wansing 2015) contains a co-implication connective that internalizes a notion of entailment different from preservation of non-truth from the conclusion of valid inferences (seen as as deductions) to the premises.

The language $\mathcal{L}_{\mathsf{2Int}}$ of 2Int is that of of BiInt, but the co-implication connective has a different meaning. In both systems, BiInt and 2Int, the intuitionistic negation $\neg A$ of A is defined as $A \to \bot$, and the co-negation $-A$ of A is defined as $\top \prec A$.

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The language $\mathcal{L}_{2\mathsf{Int}}$ of 2Int is that of of BiInt, but the co-implication connective has a different meaning. In both systems, BiInt and 2Int, the intuitionistic negation $\neg A$ of A is defined as $A \to \bot$, and the co-negation $-A$ of A is defined as $\top \prec\!\!\!\!- A$.

Definition

A model for 2Int *is a structure* $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$, *where* $\langle I, \leq \rangle$ *is a pre-order and* v^+, v^- *are functions from the set of atomic formulas to subsets of the non-empty set of states* I. *For* $x \in I$ *the relations* $\mathcal{M}, x \models^+ A$ *("x supports the truth of A in* $\mathcal{M}$*") and* $\mathcal{M}, x \models^- A$ *("x supports the falsity of A in* $\mathcal{M}$*") are inductively defined as shown below. Moreover, support of truth and support of falsity are required to be persistent. For every atomic formula* p, *and all states* x, x': *if* $x' \geq x$ *and* $\mathcal{M}, x \models^+ p$, *then* $\mathcal{M}, x' \models^+ p$ *and if* $x' \geq x$ *and* $\mathcal{M}, x \models^- p$, *then* $\mathcal{M}, x' \models^- p$.

$\mathcal{M}, x \models^+ p$ iff $x \in v^+(p)$
$\mathcal{M}, x \models^- p$ iff $x \in v^-(p)$

$\mathcal{M}, x \models^+ \top$ $\quad \mathcal{M}, x \not\models^+ \bot$
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$$\begin{array}{lll} \mathcal{M}, x \models^+ p & \text{iff} & x \in v^+(p) \\ \mathcal{M}, x \models^- p & \text{iff} & x \in v^-(p) \end{array}$$

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\mathcal{M}, x \models^{+} (A \wedge B) & \text{iff} & \mathcal{M}, x \models^{+} A \text{ and } \mathcal{M}, x \models^{+} B \\
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An $\mathcal{L}_{2\mathrm{Int}}$-formula A is dually valid in a model for 2Int $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$ iff for every $x \in I$, $\mathcal{M}, x \models^- A$ (iff for every $x \in I$, $\mathcal{M}, x \models^+ -A$); A is dually valid in 2Int ($\models^d_{2\mathrm{Int}} A$) iff A is dually valid in every model for 2Int.

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Let $\Delta \cup \{A\}$ be a set of $\mathcal{L}_{2\mathrm{Int}}$-formulas. The set Δ entails A ($\Delta \models A$) iff for every model for 2Int $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$ and every $x \in I$, it holds that if the truth of every element of Δ is supported by x, then the truth of A is supported by x.
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Now we may note that

$$A \models^{d}_{\mathsf{2Int}} B \text{ iff } \bot \models^{d}_{\mathsf{2Int}} B \prec A.$$

It is in this sense that co-implication in 2Int internalizes preservation of falsity from the premises to the conclusion of a dually valid inference.

In the following sense co-implication in 2Int is the residual of disjunction with respect to dual entailment: $A \vee B \models^{d}_{\mathsf{2Int}} C$ iff $A \models^{d}_{\mathsf{2Int}} C \prec B$ iff $B \models^{d}_{\mathsf{2Int}} C \prec A$.

Dual entailment statements express strong refutability with respect to models for 2Int.

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It is in this sense that co-implication in 2Int internalizes preservation of falsity from the premises to the conclusion of a dually valid inference.

In the following sense co-implication in 2Int is the residual of disjunction with respect to dual entailment: $A \vee B \models^d_{\mathsf{2Int}} C$ iff $A \models^d_{\mathsf{2Int}} C \prec B$ iff $B \models^d_{\mathsf{2Int}} C \prec A$.

Dual entailment statements express strong refutability with respect to models for 2Int.

The natural deduction proof system N2Int for 2Int uses single-line rules for proofs and double-line rules for dual proofs. Derivations in N2Int may contain proofs and dual proofs as subderivations.

The conclusion of a derivation therefore depends on an ordered pairs $(\Delta; \Gamma)$ of finite sets, a set of assumptions Δ and a set of counterassumptions Γ. Single square brackets [] are used to indicate that assumptions may be cancelled, and double-square brackets $[\![\ ]\!]$ are used to indicate that counterassumptions may be discharged. We write $[A]$ instead of $[\overline{A}]$ and $[\![A]\!]$ instead of $[\![\overline{\overline{A}}]\!]$.

Then we draw a distinction between rules for introducing connectives into proofs, *Ip* rules, and for eliminating them from proofs, *Ep* rule,s and rules for introducing connectives into dual proofs, *Idp* rules, and for eliminating them from dual proofs, *Edp* rules.

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The proof rules for the connectives $\top$, $\bot$, $\wedge$, $\vee$, and $\rightarrow$ are basically those of intuitionistic logic. The rules for introducing (eliminating) the connectives of intuitionistic logic into (from) dual proofs are obtained by a dualization of their introduction and elimination rules for proofs.

In 2Int the rules for introducing (eliminating) implications into (from) dual proofs are chosen in accordance with the usual understanding of the falsification conditions of implications, i.e., an implication $A \rightarrow B$ is false iff A is true and B is false. This is not the only option, see (Wansing 2008, 2015a, 2015b).

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The rules for introducing (eliminating) co-implications into (from) proofs are such that the provability of $A \prec B$ amounts to the dual provability of $A \rightarrow B$.

The introduction and elimination rules can be applied to certain basic building blocks of derivations. We consider $\overline{A}$ as a proof of A from $(\{A\};\varnothing)$ and $\overline{\overline{A}}$ as a dual proof of A from $(\varnothing;\{A\})$. Moreover $\overline{\top}$ is a proof of $\top$ from $(\varnothing;\varnothing)$ and $\overline{\overline{\bot}}$ is a dual proof of $\bot$ from $(\varnothing;\varnothing)$.

We write $(\Delta;\Gamma) \vdash A$ if there is a proof of A from $(\Delta;\Gamma)$; and we write $(\Delta;\Gamma) \vdash^d A$ if there is a dual proof of A from $(\Delta;\Gamma)$. Moreover, we assume that if $(\Delta;\Gamma) \vdash A$, $\Delta \subseteq \Delta'$ and $\Gamma \subseteq \Gamma'$ for finite sets of $\mathcal{L}_{2\mathrm{Int}}$-formulas Δ' and Γ', then $(\Delta';\Gamma') \vdash A$. Similarly, we assume that if $(\Delta;\Gamma) \vdash^d A$, $\Delta \subseteq \Delta'$ and $\Gamma \subseteq \Gamma'$ for finite sets of $\mathcal{L}_{2\mathrm{Int}}$-formulas Δ' and Γ', then $(\Delta';\Gamma') \vdash^d A$.

The system N2Int comprises the following proof rules:

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The system N2Int comprises the following proof rules:

$$\frac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\\ \hline \bot\end{matrix}}{A}\ (\bot Ep)
\qquad
\frac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\\ \hline A\end{matrix}\qquad\begin{matrix}(\Delta';\Gamma')\\ \vdots\\ \hline B\end{matrix}}{(A\wedge B)}\ (\wedge Ip)
\qquad
\frac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\\ \hline (A\wedge B)\end{matrix}}{A}\ (\wedge Ep)$$

$$\frac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\\ \hline (A\wedge B)\end{matrix}}{B}\ (\wedge Ep)
\qquad
\frac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\\ \hline A\end{matrix}}{(A\vee B)}\ (\vee Ip)
\qquad
\frac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\\ \hline B\end{matrix}}{(A\vee B)}\ (\vee Ip)$$

$$\frac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\\ \hline (A\vee B)\end{matrix}\qquad\begin{matrix}([A],\Delta';\Gamma')\\ \vdots\\ \hline C\end{matrix}\qquad\begin{matrix}([B],\Delta'';\Gamma'')\\ \vdots\\ \hline C\end{matrix}}{C}\ (\vee Ep)$$

$$\dfrac{\begin{matrix}([A], \Delta; \Gamma) \\ \vdots \\ \overline{B}\end{matrix}}{(A \rightarrow B)}\ (\rightarrow Ip) \qquad \dfrac{\begin{matrix}(\Delta; \Gamma) \\ \vdots \\ \overline{A}\end{matrix} \qquad \begin{matrix}(\Delta'; \Gamma') \\ \vdots \\ \overline{(A \rightarrow B)}\end{matrix}}{B}\ (\rightarrow Ep)$$

$$\dfrac{\begin{matrix}(\Delta; \Gamma) \\ \vdots \\ \overline{A}\end{matrix} \qquad \begin{matrix}(\Delta'; \Gamma') \\ \vdots \\ \overline{\overline{B}}\end{matrix}}{(A \prec B)}\ (\prec Ip) \qquad \dfrac{\begin{matrix}(\Delta; \Gamma) \\ \vdots \\ \overline{(A \prec B)}\end{matrix}}{A}\ (\prec Ep) \qquad \dfrac{\begin{matrix}(\Delta; \Gamma) \\ \vdots \\ \overline{(A \prec B)}\end{matrix}}{B}\ (\prec Ep)$$

Moreover, we have the following dual proof rules.

$$\dfrac{\begin{array}{c}([A], \Delta; \Gamma)\\ \vdots \\ \overline{B}\end{array}}{(A \to B)}\ (\to Ip) \qquad \dfrac{\begin{array}{c}(\Delta; \Gamma)\\ \vdots \\ \overline{A}\end{array} \quad \begin{array}{c}(\Delta'; \Gamma')\\ \vdots \\ \overline{(A \to B)}\end{array}}{B}\ (\to Ep)$$

$$\dfrac{\begin{array}{c}(\Delta; \Gamma)\\ \vdots \\ \overline{A}\end{array} \quad \begin{array}{c}(\Delta'; \Gamma')\\ \vdots \\ \overline{\overline{B}}\end{array}}{(A \mathrel{-\!\!<} B)}\ (\mathrel{-\!\!<} Ip) \qquad \dfrac{\begin{array}{c}(\Delta; \Gamma)\\ \vdots \\ \overline{(A \mathrel{-\!\!<} B)}\end{array}}{A}\ (\mathrel{-\!\!<} Ep) \qquad \begin{array}{c}(\Delta; \Gamma)\\ \vdots \\ \overline{(A \mathrel{-\!\!<} B)} \\ \overline{\overline{\phantom{(A \mathrel{-\!\!<} B)}}} \\ B\end{array}\ (\mathrel{-\!\!<} Ep)$$

Moreover, we have the following dual proof rules.

$$\dfrac{\dfrac{\begin{array}{c}(\Delta;\Gamma)\\ \vdots\end{array}}{\top}}{A}\ (\top Edp) \qquad \dfrac{\dfrac{\begin{array}{c}(\Delta;\Gamma)\\ \vdots\end{array}}{A} \quad \dfrac{\begin{array}{c}(\Delta';\Gamma')\\ \vdots\end{array}}{B}}{(A \vee B)}\ (\vee Idp) \qquad \dfrac{\dfrac{\begin{array}{c}(\Delta;\Gamma)\\ \vdots\end{array}}{(A \vee B)}}{A}\ (\vee Edp)$$

$$\dfrac{\dfrac{\begin{array}{c}(\Delta;\Gamma)\\ \vdots\end{array}}{(A \vee B)}}{B}\ (\vee Edp) \qquad \dfrac{\dfrac{\begin{array}{c}(\Delta;\Gamma)\\ \vdots\end{array}}{A}}{(A \wedge B)}\ (\wedge Idp) \qquad \dfrac{\dfrac{\begin{array}{c}(\Delta;\Gamma)\\ \vdots\end{array}}{B}}{(A \wedge B)}\ (\wedge Idp)$$

$$\dfrac{\dfrac{\overset{(\Delta;\Gamma)}{\vdots}}{(A \wedge B)} \qquad \dfrac{\overset{(\Delta;\Gamma,[\![A]\!])}{\vdots}}{C} \qquad \dfrac{\overset{(\Delta;\Gamma,[\![B]\!])}{\vdots}}{C}}{C}\ (\wedge Edp)$$

$$\dfrac{\dfrac{\overset{(\Delta;\Gamma,[\![A]\!])}{\vdots}}{B}}{(B \prec A)}\ (\prec Idp) \qquad \dfrac{\dfrac{\overset{(\Delta';\Gamma')}{\vdots}}{(B \prec A)} \qquad \dfrac{\overset{(\Delta;\Gamma)}{\vdots}}{A}}{B}\ (\prec Edp)$$

$$\dfrac{\dfrac{\overset{(\Delta;\Gamma)}{\vdots}}{A} \qquad \dfrac{\overset{(\Delta';\Gamma')}{\vdots}}{B}}{(A \to B)}\ (\to Idp) \qquad \dfrac{\dfrac{\overset{(\Delta;\Gamma)}{\vdots}}{(A \to B)}}{A}\ (\to Edp) \qquad \dfrac{\dfrac{\overset{(\Delta;\Gamma)}{\vdots}}{(A \to B)}}{B}\ (\to Edp)$$

Observation

Let $\neg\Theta := \{\neg A \mid A \in \Theta\}$ *and* $-\Theta := \{-A \mid A \in \Theta\}$ *for a set of formulas* Θ*. If* $\Theta = \varnothing$*, then* $\neg\Theta := -\Theta := \varnothing$*.*

1. $(\Delta;\Gamma) \vdash A$ *iff* $(\Delta;\Gamma) \vdash^d \neg A$*;* $(\Delta;\Gamma) \vdash^d A$ *iff* $(\Delta;\Gamma) \vdash -A$*.*
2. $(\Delta;\Gamma) \vdash A$ *iff* $(\Delta \cup -\Gamma;\varnothing) \vdash A$ *and* $(\Delta;\Gamma) \vdash^d A$ *iff* $(\varnothing;\Gamma \cup \neg\Delta) \vdash^d A$*.*

This observation reveals the difference with strong negation In D. Nelson's N4 as a switch between provability and dual provability; the back–and-forth transition between proofs and dual proofs is accomplished not by a single negation operation but by a division of labour between intuitionistic negation and co-negation:

$$\frac{\overline{A}}{-A} \qquad \frac{\overline{A}}{\overline{\overline{\neg A}}} \qquad \frac{\overline{-A}}{\overline{\overline{A}}} \qquad \frac{\overline{\overline{\neg A}}}{A}$$

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$$\begin{array}{c} \overline{\overline{A}} \\ \hline -A \end{array} \qquad \begin{array}{c} \overline{A} \\ \hline\hline \neg A \end{array} \qquad \begin{array}{c} \overline{-A} \\ \hline\hline A \end{array} \qquad \begin{array}{c} \overline{\overline{\neg A}} \\ \hline A \end{array}$$

Observation

1. N2Int *restricted to* $\mathcal{L}_{\mathsf{Int}}$, *sets of assumptions, and the Ip and Ep rules is a natural deduction proof system* NInt *for* Int.
2. *Refer to the restriction of* N2Int *to* $\mathcal{L}_{\mathsf{DualInt}}$, *sets of counterassumptions, and the Idp and Edp rules as* NDualInt. *There is an isomorphism between proofs in* NInt *and dual proofs in* NDualInt.
3. *There is an isomorphism between dual proofs in* NDualInt *and derivations in Luca Tranchini's multiple-conclusion natural deduction proof system for DualInt.*

The proof system N2Int is sound and complete with respect to its relational semantics.

Theorem

Let A be an $\mathcal{L}_{2\mathrm{Int}}$-formula and let $\{A_1, \dots, A_k\}$, $\{B_1, \dots, B_m\}$ be finite, possibly empty sets of $\mathcal{L}_{2\mathrm{Int}}$-formulas.

1. $(\{A_1, \dots, A_k\}; \{B_1, \dots, B_m\}) \vdash A$ *iff* $\{A_1, \dots, A_k, -B_1, \dots, -B_m\} \models A$;
2. $(\{A_1, \dots, A_k\}; \{B_1, \dots, B_m\}) \vdash^d A$ *iff* $\{\neg A_1, \dots, \neg A_k, B_1, \dots, B_m\} \models^d A$.

Definition

Let $\Phi' = \{p' \mid p \in \Phi\}$. The translation τ from $\mathcal{L}_{2\mathsf{Int}}$ into $\mathcal{L}_{\mathsf{Int}}$ based on the set of atomic formulas $\Phi \cup \Phi'$ is defined as follows :

$$
\begin{array}{rcl@{\qquad}rcl}
\tau(p) & := & p & \tau(-p) & = & p' \\
\tau(\top) & := & \top & \tau(-\top) & := & \bot \\
\tau(\bot) & := & \bot & \tau(-\bot) & := & \top \\
\tau(A \wedge B) & := & \tau(A) \wedge \tau(B) & \tau(-(A \wedge B)) & := & \tau(-A) \vee \tau(-B) \\
\tau(A \vee B) & := & \tau(A) \vee \tau(B) & \tau(-(A \vee B)) & := & \tau(-A) \wedge \tau(-B) \\
\tau(A \to B) & := & \tau(A) \to \tau(B) & \tau(-(A \to B)) & := & \tau(A) \wedge \tau(-B) \\
\tau(A \prec B) & := & \tau(A) \wedge \tau(-B), & \tau(-(A \prec B)) & := & \tau(-B) \to \tau(-A) \\
 & & \text{if } A \not\equiv \top & & &
\end{array}
$$

Theorem

Let A be any formula from $\mathcal{L}_{2\mathsf{Int}}$. Then $\models_{2\mathsf{Int}} A$ iff $\models_{\mathsf{Int}} \tau(A)$.

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Theorem

For every $\mathcal{L}_{\mathbf{Int}}$-formula A, $\models_{\mathbf{Int}} A$ iff $\models_{\mathbf{2Int}} A$.

There is an analogous translation ζ from $\mathcal{L}_{\mathbf{2Int}}$ into $\mathcal{L}_{\mathbf{DualInt}}$

Theorem

For every $\mathcal{L}_{\mathbf{2Int}}$-formula A, $\models^{d}_{\mathbf{2Int}} A$ iff $\models^{d}_{\mathbf{DualInt}} \zeta(A)$.

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For every $\mathcal{L}_{\mathbf{DualInt}}$-formula A, $\models^{d}_{\mathbf{DualInt}} A$ iff $\models^{d}_{\mathbf{2Int}} A$.

Normal derivations in N2Int

Detour conversions

In addition to the intuitionistic detour (or β) conversions for proofs there are now detour conversions for dual proofs. Let $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}_1$, $\mathcal{D}_2$ stand for derivations in N2Int. The derivations on the left hand side of $\rightsquigarrow$ are converted into the derivations on the right hand side of $\rightsquigarrow$; $i \in \{1, 2\}$:

$\wedge$-conversions:

$$\cfrac{\cfrac{\cfrac{\mathcal{D}_1}{A_1} \qquad \cfrac{\mathcal{D}_2}{A_2}}{A_1 \wedge A_2}}{A_i} \quad \rightsquigarrow \quad \cfrac{\mathcal{D}_i}{A_i} \qquad\qquad \cfrac{\cfrac{\cfrac{\mathcal{D}}{A_i}}{A_1 \wedge A_2} \qquad \cfrac{\begin{matrix}[\![A_1]\!]\\ \mathcal{D}_1\end{matrix}}{C} \qquad \cfrac{\begin{matrix}[\![A_2]\!]\\ \mathcal{D}_2\end{matrix}}{C}}{C} \quad \rightsquigarrow \quad \cfrac{\begin{matrix}\cfrac{\mathcal{D}}{A_i}\\ \mathcal{D}_i\end{matrix}}{C}$$

$\vee$-conversions:

$$\begin{array}{c}\begin{array}{c}\begin{array}{c}\mathcal{D}\\ \hline A_i\end{array}\\ \hline A_1 \vee A_2\end{array} \quad \begin{array}{c}[A_1]\\ \mathcal{D}_1\\ \hline C\end{array} \quad \begin{array}{c}[A_2]\\ \mathcal{D}_2\\ \hline C\end{array}\\ \hline C\end{array} \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{D}\\ \hline A_i\\ \mathcal{D}_i\\ \hline C\end{array} \qquad\qquad \begin{array}{c}\begin{array}{c}\mathcal{D}_1\\ \hline\hline A_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \hline\hline A_2\end{array}\\ \hline\hline A_1 \vee A_2\\ \hline\hline A_i\end{array} \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{D}_i\\ \hline\hline A_i\end{array}$$

$\rightarrow$-conversions:

$$\begin{array}{c}\begin{array}{c}[A]\\ \mathcal{D}\\ \hline B\\ \hline A \rightarrow B\end{array} \quad \begin{array}{c}\mathcal{D}_1\\ \hline A\end{array}\\ \hline B\end{array} \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{D}_1\\ \hline A\\ \mathcal{D}\\ \hline B\end{array} \qquad\qquad \begin{array}{c}\begin{array}{c}\mathcal{D}_1\\ \hline A\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \hline\hline B\end{array}\\ \hline\hline A \rightarrow B\\ \hline A\end{array} \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{D}_1\\ \hline A\end{array}$$

$\mathrel{-\!\!<}$-conversions:

$$\AxiomC{$\mathcal{D}_1$}\UnaryInfC{A}\AxiomC{$\mathcal{D}_2$}\doubleLine\UnaryInfC{B}\BinaryInfC{$A \mathrel{-\!\!<} B$}\UnaryInfC{A}\DisplayProof \quad \rightsquigarrow \quad \AxiomC{$\mathcal{D}_1$}\UnaryInfC{A}\DisplayProof \qquad\qquad \AxiomC{$\mathcal{D}_1$}\UnaryInfC{A}\AxiomC{$\mathcal{D}_2$}\doubleLine\UnaryInfC{B}\BinaryInfC{$A \mathrel{-\!\!<} B$}\doubleLine\UnaryInfC{B}\DisplayProof \quad \rightsquigarrow \quad \AxiomC{$\mathcal{D}_2$}\doubleLine\UnaryInfC{B}\DisplayProof$$

Permutation conversions

Depending on whether $\vee$-eliminations from proofs or $\wedge$-introductions into dual proofs are permuted over eliminations from proofs or dual proofs, we get four different kinds of permutation conversions.

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{A \vee B} \quad \dfrac{\mathcal{D}_1}{C} \quad \dfrac{\mathcal{D}_2}{C}}{C} \qquad \mathcal{D}'}{D}\ \textit{Ep}\text{ rule} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\mathcal{D}}{A \vee B} \quad \dfrac{\dfrac{\mathcal{D}_1}{C} \quad \mathcal{D}'}{D} \quad \dfrac{\dfrac{\mathcal{D}_2}{C} \quad \mathcal{D}'}{D}}{D}$$

$$\begin{array}{c} \begin{array}{c} \begin{array}{c}\mathcal{D} \\ \hline\hline A \wedge B\end{array} \quad \begin{array}{c}\mathcal{D}_1 \\ \hline\hline C\end{array} \quad \begin{array}{c}\mathcal{D}_2 \\ \hline\hline C\end{array} \\ \hline\hline C \end{array} \qquad \mathcal{D}' \\ \hline\hline D \end{array}\ \textit{Edp}\text{ rule} \qquad \rightsquigarrow \qquad \begin{array}{c} \begin{array}{c}\mathcal{D} \\ \hline\hline A \wedge B\end{array} \quad \begin{array}{c} \begin{array}{c}\mathcal{D}_1 \\ \hline\hline C\end{array} \quad \mathcal{D}' \\ \hline\hline D \end{array} \quad \begin{array}{c} \begin{array}{c}\mathcal{D}_2 \\ \hline\hline C\end{array} \quad \mathcal{D}' \\ \hline\hline D \end{array} \\ \hline\hline D \end{array}$$

Simplification conversions

Next to simplification conversions arising from $\vee$-eliminations (from proofs) in which no assumptions are discharged, we also consider simplification conversions arising from $\wedge$-introductions (into dual proofs) in which no counterassumptions are cancelled.

$$\dfrac{\dfrac{\mathcal{D}}{A \vee B} \qquad \dfrac{\mathcal{D}_1}{C} \qquad \dfrac{\mathcal{D}_2}{C}}{D} \quad \rightsquigarrow \quad \dfrac{\mathcal{D}_i}{C} \qquad\qquad \begin{array}{c} \begin{array}{c} \mathcal{D} \\ \hline\hline A \wedge B \end{array} \qquad \begin{array}{c} \mathcal{D}_1 \\ \hline\hline C \end{array} \qquad \begin{array}{c} \mathcal{D}_2 \\ \hline\hline C \end{array} \\ \hline\hline D \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \mathcal{D}_i \\ \hline\hline C \end{array}$$

where no assumptions are cancelled by $\vee Ep$ or $\wedge Edp$ in $\mathcal{D}_i$ for $i \in \{1, 2\}$.

We refer to the relation defined by the above conversions that exhibit only proofs (dual proofs) as $\rightsquigarrow_{\mathsf{Int}}$ ($\rightsquigarrow_{\mathsf{DualInt}}$).

Observation

Let $\mathcal{D}$, $\mathcal{D}'$ be derivations in N2Int*. If $\mathcal{D} \rightsquigarrow_{\mathsf{Int}} \mathcal{D}'$, then $\delta(\mathcal{D}) \rightsquigarrow_{\mathsf{DualInt}} \delta(\mathcal{D}')$.*

Definition

A derivation in N2Int *is in normal form iff there is no derivation to which it can be converted.*

Theorem

For every derivation of a formula A in N2Int from a pair $(\Delta; \Gamma)$ of finite sets of assumptions and counterassumptions there exists a normal derivation of A from $(\Delta; \Gamma)$.

We refer to the relation defined by the above conversions that exhibit only proofs (dual proofs) as $\rightsquigarrow_{\mathsf{Int}}$ ($\rightsquigarrow_{\mathsf{DualInt}}$).

Observation

Let $\mathcal{D}$, $\mathcal{D}'$ be derivations in N2Int*. If $\mathcal{D} \rightsquigarrow_{\mathsf{Int}} \mathcal{D}'$, then $\delta(\mathcal{D}) \rightsquigarrow_{\mathsf{DualInt}} \delta(\mathcal{D}')$.*

Definition

A derivation in N2Int *is in normal form iff there is no derivation to which it can be converted.*

Theorem

For every derivation of a formula A in N2Int from a pair $(\Delta; \Gamma)$ of finite sets of assumptions and counterassumptions there exists a normal derivation of A from $(\Delta; \Gamma)$.

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Some results by Sergey Drobyshevich

In a recent paper, Sergey Drobyshevich, has presented a bilateral Hilbert-style calculus, H2Int, for 2Int defined over signed formulas of two types: formulas signed with plus intuitively correspond to assertion, while formulas signed with minus correspond to rejection (denial).

The axioms of H2Int are:

- all intuitionistic axioms in the language of intuitionistic logic with plus signs;
- their duals with minus signs;
- plus the following axioms:

$(s1^+)$ $(A \prec B) \leftrightarrow (A \wedge -B)^+$,
$(s1^-)$ $(A \to B) \succ\!\prec (A \vee \neg B)^-$,
$(s2^+)$ $-(A \to B) \leftrightarrow (A \wedge -B)^+$,
$(s2^-)$ $\neg(A \prec B) \succ\!\prec (A \vee \neg B)^-$,
$(s3^+)$ $-(A \prec B) \to -(B \to -A)^+$,
$(s3^-)$ $(\neg A \prec \neg B) \prec \neg(B \to A)^-$.

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The rules of inference of H2Int are:

$$\frac{A^{+} \quad (A \rightarrow B)^{+}}{B^{+}}\ (mp) \qquad \frac{(B \prec A)^{-} \quad A^{-}}{B^{-}}\ (dmp)$$

and the following interaction rules:

$$\frac{A^{+} \quad B^{-}}{(A \prec B)^{+}}\ (r1^{+}) \qquad \frac{(A \prec B)^{+}}{B^{-}}\ (r2)^{+}$$

$$\frac{A^{+} \quad B^{-}}{(A \rightarrow B)^{-}}\ (r1^{-}) \qquad \frac{(A \rightarrow B)^{-}}{A^{+}}\ (r2)^{-}$$

A derivability relation $\vdash^{s}_{\mathrm{H2Int}}$ is defined as usual, but now for signed formulas.

Then let 2Int$i$ be the axiomatic extension of intuitionistic logic with the non-signed versions of axioms $(s1^{+})$ – $(s3^{+})$ and let $\vdash^{i}_{\mathrm{2Int}}$ be its derivability relation.

Theorem

1. *There are no formulas* $(A \prec B)$ *such that* $\vdash^{i}_{\mathrm{2Int}} (A \prec B)$.
2. *There are formulas* $(A \prec B)$ *such that* $\vdash^{s}_{\mathrm{H2Int}} (A \prec B)^{+}$

The theorem shows that the interplay rules (and hence the minus-signed fragment of 2Int) are indeed involved in deriving plus-signed formulas.

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Conclusion

- There is more than one notion of refutability.
- In a four-valued semantics it makes sense to distinguish between entailment and dual entailment: strong refutability.
- If strong refutability is meant to characterize a constructive inference relation that is independent from a constructive inference relation characterized by valid entailment, dual derivability in 2Int is such a refutability relation.

- Sergey Drobyshevich, A Hilbert-style calculus that takes rejection seriously, 2018, submitted.
- Tomasz Skura, A Refutation Theory, *Logica Universalis* 3 (2009), 293–302.
- Tomasz Skura, What Is a Refutation System?, in: A. Moktefi, A. Moretti, and F. Shang (eds.), *Let's Be Logical*, College Publications, London, 189–198.
- Heinrich Wansing, "Falsification, natural deduction and bi-intuitionistic logic, *Journal of Logic and Computation*, doi: 10.1093/logcom/ext035, first published online: July 17, 2013.
- Heinrich Wansing, "On split negation, strong negation, information, falsification, and verification, 2015, to appear in: K. Bimbó (ed.), *J. Michael Dunn on Information Based Logics*, Outstanding Contributions to Logic Series, Springer.
- Heinrich Wansing, A more general general proof theory, *Journal of Applied Logic* 25 (2017), 23–46.