A refutation calculus for intuitionistic logic

(and a proof calculus for dual-intuitionistic logic)

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$$\Gamma \qquad \vdash \qquad \Delta$$

If all

$\Gamma \qquad \vdash \qquad \Delta$

are true

If all		at least one
Γ	$\vdash$	Δ
are true		must be true

$$\Gamma \quad \vdash \quad \Delta$$

If all

$\Gamma \quad \vdash \quad \Delta$

are false

at least one		If all
Γ	$\vdash$	Δ
must be false		are false

$$\Gamma \quad \vdash \quad \Delta$$

Constructivism:
make “transmission” precise

Intuitionism
(and its proofs)

$$\Gamma \qquad \vdash \qquad B$$

If all

$\Gamma \qquad \vdash \qquad B$

are provable

If all

$\Gamma \vdash B$

are provable

must be provable

$\overrightarrow{x} : \Gamma \qquad \vdash \qquad B$

must be provable

$$\vec{x} : \Gamma \vdash t : B$$

What is a proof of a logically complex sentence?

- a proof of $A \wedge B$ is a pair
  consisting of a proof of A and of a proof of B
- a proof of $A \vee B$ is a proof of either A or B
  with a bit of information telling which of the two
- a proof of $A \supset B$ is a function
  from proofs of A to proofs of B
- …

Intuitionistic Logic: $\mathtt{NI}^{\wedge\supset}$

$$\frac{t : A \qquad s : B}{\langle t, s \rangle : A \wedge B}\ \wedge I$$

$$\frac{t : A \wedge B}{\pi_1(t) : A}\ \wedge E_1$$

$$\frac{t : A \wedge B}{\pi_2(t) : B}\ \wedge E_2$$

$$\frac{\begin{array}{c}[x : A] \\ t : B\end{array}}{\lambda x.t : A \supset B}\ \supset I$$

$$\frac{t : A \supset B \qquad s : A}{\mathsf{app}(t, s) : B}\ \supset E$$

$$y : B \quad \vdash \quad \lambda x.\langle x, y\rangle : A \supset (A \wedge B)$$

$$\dfrac{\dfrac{\cancel{x : A} \qquad y : B}{\langle x, y\rangle : A \wedge B}}{\lambda x.\langle x, y\rangle : A \supset (A \wedge B)}$$

Dual-Intuitionism
(and its refutations)

$$A \quad \vdash \quad \Delta$$

If all

$A \quad \vdash \quad \Delta$

are refutable

If all

$A \vdash \Delta$

must be refutable

are refutable

$$A \quad \vdash \quad \overrightarrow{x} : \Delta$$

must be refutable

$$t : A \quad \vdash \quad \overrightarrow{x} : \Delta$$

What is a refutation of a logically complex sentence?

- a refutation of $A \wedge B$ is a refutation of either A or B with a bit of information telling which of the two
- a refutation of $A \vee B$ is a pair consisting of two refutations, of A and of B
- a refutation of $B \not\subset A$ is a function from refutations of A to refutations of B
- ...

Dual-intuitionistic Logic: $\mathtt{NDI}^{\vee\not\subset}$

$$\frac{\pi_1(t) : A}{t : A \vee B}\ \vee I_1 \qquad \frac{\langle t, s\rangle : A \vee B}{t : A \qquad s : B}\ \vee E$$

$$\frac{\pi_2(t) : B}{t : A \vee B}\ \vee I_2$$

$$\frac{\mathsf{app}(t, s) : B}{t : B \not\subset A \qquad s : A}\ \not\subset I \qquad \frac{\lambda x.t : B \not\subset A}{\begin{array}{c} t : B \\ [x : A] \end{array}}\ \not\subset E$$

$$\lambda x.\langle x, y\rangle : (A \vee B) \notin A \quad \vdash \quad y : B$$

$$\dfrac{\lambda x.\langle x, y\rangle : (A \vee B) \notin A}{\dfrac{\langle x, y\rangle : A \vee B}{\cancel{x : A} \qquad y : B}}$$

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- Nelson-Wansing style:

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  - a refutation of $A \supset B$ is a pair
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- Our Proposal:
  Refutations and proofs independently defined

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$$\Gamma \vdash x : B$$

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$$\vec{t} : \Gamma \vdash x : B$$

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$$\frac{t \quad : A \wedge B}{\pi_1(t) : A}\ \wedge E_1 \qquad\qquad \frac{t \quad : A \wedge B}{\pi_2(t) : B}\ \wedge E_2$$

$$\frac{? \quad : A \wedge B}{t \quad : A} \wedge E_1 \qquad\qquad \frac{? \quad : A \wedge B}{t \quad : B} \wedge E_2$$

$$\frac{\mathtt{in}_1(t) : A \wedge B}{t \;\; : A} \wedge E_1 \qquad\qquad \frac{\mathtt{in}_2(t) : A \wedge B}{t \;\; : B} \wedge E_2$$

$$\dfrac{t \quad : A \qquad\qquad s \quad : B}{\langle t, s \rangle : A \wedge B}\ \wedge\text{I}$$

$$\frac{\color{blue}{?} : A \qquad \color{blue}{?} : B}{\color{blue}{t} : A \wedge B} \wedge I$$

$$\frac{\mathsf{case}_1(\mathsf{t}) : A \qquad \mathsf{case}_2(\mathsf{t}) : B}{\mathsf{t} \;: A \wedge B} \wedge\mathrm{I}$$

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  the hypothesis that it is impossible to
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  the *hypothesis* that it is *impossible* to
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$$\frac{t \quad : A \supset B \qquad\qquad s \quad : A}{\mathsf{app}(t, s) : B}\ \supset\!E$$

$$\frac{\;? \; : A \supset B \qquad\qquad ? \; : A\;}{t \; : B} \supset\!E$$

$$\dfrac{\mathtt{jump}\ (t, x) : A \supset B \qquad\qquad \mathbf{x}_t : A}{t \ : B}\ {\supset}\mathrm{E}$$

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$$\dfrac{\begin{array}{c}[\,x : A\,] \\ t : B\end{array}}{\lambda x.t : A \supset B}\ {\supset}\mathrm{I}$$

$$\dfrac{\begin{array}{c}[\,?\;:A\,]\\ ?\;:B\end{array}}{t\;:A \supset B}\supset\mathrm{I}$$

$$\dfrac{\begin{array}{c}[\,?\;:A]\\ \mathtt{x}_t : B\end{array}}{t \;: A \supset B}\supset\!\mathrm{I}$$

$$\dfrac{\begin{array}{c}[\,s : A\,] \\ \mathbf{x}_t : B\end{array}}{t : A \supset B}\ \supset I$$

$$\dfrac{\begin{array}{c} t \\ \boxed{x \mapsto s^*} \\ [\, s : A \,] \\ \mathbf{x}_t : B \end{array}}{t : A \supset B}\ \supset\!\mathrm{I}$$

$$s^* = s[\![x/\mathbf{x}_t]\!]$$

$$\overset{x}{\boxed{y \mapsto \mathtt{case}_1(y)}}\;,\; \mathtt{case}_2(\mathtt{y}_x) : B \;\vdash\; x : A \supset (A \wedge B)$$

$$\dfrac{\dfrac{\overset{x}{\boxed{y \mapsto \mathtt{case}_1(y)}} \atop \mathtt{case}_1(\mathtt{y}_x) : A \qquad \mathtt{case}_2(\mathtt{y}_x) : B}{\mathtt{y}_x : A \wedge B}}{x : A \supset (A \wedge B)}$$

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What is a proof of a logically complex sentence?

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- a proof of $B \nleftarrow A$ is a pair consisting of a proof of B and a refutation of A

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$\vee$

$$\frac{\pi_1(t) : A}{t : A \vee B}\ \vee I_1$$

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$$\frac{t : A}{\mathtt{in}_1(t) : A \vee B}\ \vee I_1$$

$$\frac{t : A \vee B}{\mathtt{case}_1(t) : A \qquad \mathtt{case}_2(t) : B}\ \vee E$$

$$\frac{t : B}{\mathtt{in}_2(t) : A \vee B}\ \vee I_2$$

$\not\subset$

$$\frac{\mathsf{app}(t, s) : A}{t : A \not\subset B \qquad s : B}\;{\not\subset}\mathrm{I}$$

$$\frac{\lambda x.t : A \not\subset B}{\begin{array}{c} t : A \\ [x : B] \end{array}}\;{\not\subset}\mathrm{E}$$

$$\frac{t : B}{\mathtt{jump}(t, x) : B \not\subset A \qquad \mathbf{x}_t : A}\;{\not\subset}\mathrm{I}$$

$$\frac{t : B \not\subset A}{\begin{array}{c} \mathbf{x}_t : B \\ [s : A] \\ \boxed{x \mapsto s[\![x/\mathbf{x}_t]\!]} \\ t \end{array}}\;{\not\subset}\mathrm{E}$$

- Prawitz: Intuitionistic proofs

transform justifications of assertions
into
justification of assertions

- Can we say that dual-intuitionistic proofs

transform justification of hypothesis
into
justification of hypothesis?

(Shramko, Bellin)

Bi-intuitionistic Logic

$$
\begin{array}{ccc}
 & A \vee D & \\
 & \overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \; \vee E & \\
A \supset B & A \qquad\qquad\qquad\qquad\qquad\qquad D & \\
\multicolumn{2}{c}{\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \; \supset E \quad \overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \; \not\in I} & \\
 & B \qquad\qquad\qquad\qquad C \qquad\qquad\qquad\qquad\qquad D \not\in C &
\end{array}
$$

$$y : A \supset B, x : A \vee D$$

$$\vdash$$

$$\mathtt{app}(y, \mathtt{case}_1(x)) : B, \mathtt{z}_{\mathtt{case}_2(x)} : C, \mathtt{jump}(\mathtt{case}_2(x), z) : D \notin C$$

$$\dfrac{A \vee D}{A \qquad\qquad D}\ \vee E$$

$$\dfrac{A \supset B \qquad\qquad A}{B}\ \supset E \qquad \dfrac{D}{C \qquad\qquad D \notin C}\ \notin I$$

$$y : A \supset B, x : A \vee D$$

$$\vdash$$

$$\mathtt{app}(y, \mathtt{case}_1(x)) : B, \mathtt{z}_{\mathtt{case}_2(x)} : C, \mathtt{jump}(\mathtt{case}_2(x), z) : D \not\subset C$$

$$\dfrac{y : A \supset B \qquad \dfrac{x : A \vee D}{\mathtt{case}_1(x) : A \qquad\qquad \mathtt{case}_2(x) : D}\ \vee E}{\mathtt{app}(y, \mathtt{case}_1(x)) : B}\ \supset E \qquad \dfrac{}{\mathtt{z}_{\mathtt{case}_2(x)} : C \qquad \mathtt{jump}(\mathtt{case}_2(x), z) : D \not\subset C}\ \not\subset I$$

$$\mathtt{jump}(x, y) : A \supset B, \langle \mathtt{y}_x, \mathtt{app}(w, z)\rangle : A \vee D$$

$$\vdash$$

$$x : B, w : C, z : D \not\subset C$$

$$y : A \supset B, x : A \vee D$$

$$\vdash$$

$$\mathtt{app}(y, \mathtt{case}_1(x)) : B, \mathtt{z}_{\mathtt{case}_2(x)} : C, \mathtt{jump}(\mathtt{case}_2(x), z) : D \notin C$$

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$$\mathtt{jump}(x, y) : A \supset B, \langle \mathtt{y}_x, \mathtt{app}(w, z) \rangle : A \vee D$$

$$\vdash$$

$$x : B, w : C, z : D \notin C$$

$$y : A \supset B, x : A \vee D \vdash \mathtt{app}(y, \mathtt{case}_1(x)) : B, \mathtt{z}_{\mathtt{case}_2(x)} : C, \mathtt{jump}(\mathtt{case}_2(x), z) : D \not\in C$$

$$\dfrac{\mathtt{jump}(x,y) : A \supset B \qquad \dfrac{\langle \mathtt{y}_x, \mathtt{app}(w,z) \rangle : A \vee D}{\mathtt{y}_x : A \qquad\qquad \mathtt{app}(w,z) : D}\,\vee E}{x : B \quad \supset E \qquad\qquad w : C \qquad\qquad z : D \not\in C \quad \not\in I}$$

$$\mathtt{jump}(x,y) : A \supset B, \langle \mathtt{y}_x, \mathtt{app}(w,z) \rangle : A \vee D \vdash x : B, w : C, z : D \not\in C$$

Thanks for your attention!