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IMPLICATIONAL LOGIC, RELEVANCE, AND REFUTABILITY

Abstract. The goal of this paper is to analyse Implicational Relevance Logic from the point of view of refutability. We also correct an inaccuracy in our paper "The RM paraconsistent refutation system" (DOI: 10.12775/LLP.2009.005).

Keywords: implicational logic; relevance logic; refutation systems

1. Introduction

Propositional logics are usually motivated by positive properties and conditions. However, negative motivations are also possible. A prime example here is Paraconsistent Logic, which is obtained from Classical Logic by rejecting the law of explosion. Another natural example is Implicational Relevance Logic, in which the intuitionistic law

$$p \to (q \to p) \tag{P}$$

is rejected. (P) (or Positive Paradox) says that a true proposition is entailed by anything; so, of course, it is not acceptable for relevance logicians. Church's axioms for $\mathbf{R}_{\to}$ (the implicational fragment of $\mathbf{R}$) can be viewed as obtained from those for $\mathbf{H}_{\to}$ (the implicational fragment of Intuitionistic Logic) by taking $p \to p$ instead of (P) [see 5].

If we assume that the meaning of the connective $\to$ is motivated by the concept of (constructive) proof, then it is natural to require that Implicational Relevance Logic should be a (proper) part of $\mathbf{H}_{\to}$. In other words, every formula that is not in $\mathbf{H}_{\to}$ should be rejected.

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Yet another example is the variable-sharing property (*VSP* for short), which can be presented as a negative property:

$A \to B$ is rejected, whenever A and B share no variable.

We will show that, in a large class of implicational logics, *VSP* is equivalent to the simple property that (P) is rejected.

Our *non-negative approach* can be outlined as follows. Let $\boldsymbol{L}$ be a logic (that is, a set of formulas closed under *substitution*, *modus ponens*, and possibly some other rules), and let *NEG* be a set of formulas that we want to reject. We present $\boldsymbol{L}$ as an axiomatic system consisting of the inference rules together with some acceptable axioms $POS \subseteq \boldsymbol{L}$ in such a way that no $A \in NEG$ is derivable from *POS*.

In the standard positive approach, the new logic is the set of *provable* formulas; that is formulas derivable from *POS* by the rules. Our non-standard non-negative approach is different. We keep *NEG*, and we declare the formulas in *NEG rejected* (or "refutation axioms"). We then say that a formula A is *refutable* iff some $B \in NEG$ is derivable from A by using acceptable axioms (and rules). We have thus defined the set $Ref(POS, NEG)$ of refutable formulas. If the complement $\boldsymbol{L}^*$ $(:= -Ref(POS, NEG))$ of this set is closed under the inference rules, then $\boldsymbol{L}^*$ is our new logic disjoint with *NEG*.

In a nutshell: in the positive approach, we want what is good; and in the non-negative approach, we prevent what is bad. These are the two extremes of possible solutions.

In this paper, we analyse Implicational Relevance Logic from the point of view of refutability. As acceptable axioms we take those of the relevance logic $\mathbf{RMO}^{\top}_{\to}$ ($\mathbf{RMO}_{\to}$ together with the axiom $p \to \top$ for the constant $\top$, which proves useful here). Our refutation axiom is (P) together with $-\mathbf{H}_{\to}$. It turns out that the resulting logic $\mathbf{H}^*_{\to}$ is the greatest extension of $\mathbf{RMO}^{\top}_{\to}$ that is weaker than $\mathbf{H}_{\to}$.

We also correct an inaccuracy in the paper [10].

2. Implicational logic and relevance

The meaning of the intuitionistic connective $\to$ is determined by the STANDARD DEDUCTION THEOREM [see, e.g., 1, 5]:

$\vdash A_1 \to (A_2 \to \cdots \to (A_n \to B) \dots)$ *iff there is a deduction of* B *from* $A_1, \dots, A_n$.

Now, the meaning of the relevant connective $\to$ is provided by the RELEVANT DEDUCTION THEOREM (a modification of the above) [see 1, 5]:

$\vdash A_1 \to (A_2 \to \cdots \to (A_n \to B)\ldots)$ *iff there is a deduction of* B *from* $A_1, \ldots, A_n$*, in which all members of* $\{A_1, \ldots, A_n\}$ *are used.*

For example, here is a deduction for (P).

1. $\mid p$ hyp
2. $\mid \mid q$ hyp
3. $\mid \mid p$ 1, reit
4. $\mid q \to p$ 2, 3, $\to$I
5. $p \to (q \to p)$ 1, 4, $\to$I

Hence (P) is a valid principle of $\mathbf{H}_\to$. Note that q is not used in the above deduction.

Following Avron [3, 4], we regard $\{A_1, \ldots, A_n\}$ as a set. Thus, the mingle axiom $p \to (p \to p)$ has a relevant deduction.

Of course, the $\mathbf{R}_\to$ axioms also have relevant deductions, so we get the $\mathbf{RMO}_\to$ axioms as our acceptable axioms.

Note that if you view $\{A_1, \ldots, A_n\}$ as a multiset, then $p \to (p \to p)$ is not acceptable and you must replace it with $p \to p$, obtaining the $\mathbf{R}_\to$ axioms rather than the $\mathbf{RMO}_\to$ ones.

3. The logic $\mathbf{RMO}^\top_\to$

Let *For* be the set of all formulas generated from the set

$$Var = \{p, q, r, p_1, p_2, \ldots\}$$

by the connective $\to$ and the constant $\top$ (usually denoted by T). By a *substitution* we mean a function s from *Var* to *For* extended to all formulas as follows:

$$s(\top) = \top \quad \text{and} \quad s(A \to B) = s(A) \to s(B).$$

We say that a set $\boldsymbol{X}$ of formulas is *closed under substitution* iff $s(A) \in \boldsymbol{X}$ whenever $A \in \boldsymbol{X}$. Moreover, we say that a set $\boldsymbol{X}$ of formulas is *closed under modus ponens* iff $B \in \boldsymbol{X}$ whenever $A \in \boldsymbol{X}$ and $A \to B \in \boldsymbol{X}$. We note that for all $\boldsymbol{X}, \boldsymbol{Y} \subseteq For$:

- If $\boldsymbol{X}, \boldsymbol{Y}$ are closed under substitution (resp. modus ponens), then so is $\boldsymbol{X} \cap \boldsymbol{Y}$.

The logic $\mathbf{RMO}_{\to}^{\top}$ is the smallest set of formulas closed under substitution and modus ponens, and containing the following axioms [see, e.g., 5]:

$$p \to (p \to p) \tag{A1}$$

$$(p \to q) \to ((q \to r) \to (p \to r)) \tag{A2}$$

$$(p \to (q \to r)) \to (q \to (p \to r)) \tag{A3}$$

$$(p \to (p \to q)) \to (p \to q) \tag{A4}$$

$$p \to \top \tag{A5}$$

We also write $\vdash A$ instead of $A \in \mathbf{RMO}_{\to}^{\top}$. Notice that the following formulas belong to $\mathbf{RMO}_{\to}^{\top}$:

$$p \to p \tag{C1}$$

$$p \to ((p \to q) \to q) \tag{C2}$$

$$(p \to q) \to ((r \to p) \to (r \to q)) \tag{C3}$$

$$((p \to p) \to q) \to q \tag{C4}$$

$$(p \to (q \to r)) \to ((p \to q) \to (p \to r)) \tag{C5}$$

$$(p_1 \to (p \to q)) \to ((p_1 \to (q \to r)) \to (p_1 \to (p \to r))) \tag{C6}$$

$$(p \to q) \to (p \to (p \to q)) \tag{C7}$$

$$(\top \to (p \to p)) \to (p \to (q \to p)) \tag{C8}$$

PROOF. For (C1):

1. $p \to (p \to p)$ — (A1)
2. $p \to p$ — 1, (A4), *mp*

For (C2):

1. $(p \to q) \to (p \to q)$ — (A1)
2. $p \to ((p \to q) \to q)$ — 1, (A3), *mp*

For (C3):

1. $(r \to p) \to ((p \to q) \to (r \to q))$ — (A2)
2. $(p \to q) \to ((r \to p) \to (r \to q))$ — 1, (A3), *mp*

For (C4): By (C1), (C2), *mp*.

For (C5):

1. $(p \to q) \to ((q \to (p \to r)) \to (p \to (p \to r)))$ — (A2)
2. $(p \to q) \to ((q \to (p \to r)) \to (p \to r))$ — 1, (A2), (A4), *mp*
3. $(q \to (p \to r)) \to ((p \to q) \to (p \to r))$ — 2, (A3), *mp*
4. $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ — 3, (A2), (A3), *mp*

For (C6): By (A2), (C5), *mp*.

For (C7):

1. $(p \to q) \to (p \to (q \to q))$ (A1), (A2), (A3), *mp*
2. $(p \to q) \to ((q \to q) \to (p \to q))$ (A2)
3. $(p \to q) \to (p \to (p \to q))$ 1, 2, (C6), *mp*

For (C8):

1. $q \to \top$ (A5)
2. $(\top \to (p \to p)) \to (q \to (p \to p))$ 1, (A2), *mp*
3. $(\top \to (p \to p)) \to (p \to (q \to p))$ 2, (A2), (A3), *mp* □

4. Refutability

PROPOSITION 4.1. $\mathbf{H}_\to$ *is the axiomatic strengthening of* $\mathbf{RMO}^\top_\to$ *by* (P).

PROOF. Let $\mathbf{RMO}^{\top(\mathtt{P})}_\to$ be the least set containing $\mathbf{RMO}^\top_\to \cup \{(\mathtt{P})\}$ and closed under substitution and *mp*. Then $\mathbf{H}_\to \subseteq \mathbf{RMO}^{\top(\mathtt{P})}_\to$ (because (P), C5 are in $\mathbf{RMO}^{\top(\mathtt{P})}_\to$). Also, both (P) and the axioms of $\mathbf{RMO}^\top_\to$ are in $\mathbf{H}_\to$, so $\mathbf{RMO}^{\top(\mathtt{P})}_\to \subseteq \mathbf{H}_\to$, which gives the result. □

Note that $\top$ is redundant in $\mathbf{H}_\to$, because both $(p \to p) \to \top$ and $\top \to (p \to p)$ (by (A3) and (P)) are in $\mathbf{H}_\to$.

We define the matrix $\mathfrak{3} := (\{-1, 0, 1\}, \{0, 1\}, \to)$, where [see 7]:

$$x \to y = \begin{cases} \max(-x, y) & \text{if } x \leq y, \\ \min(-x, y) & \text{otherwise.} \end{cases}$$

A *valuation in* $\mathfrak{3}$ is a function v from *Var* to $\{-1, 0, 1\}$ extended as follows:

$$v(\top) = 1 \quad \text{and} \quad v(A \to B) = v(A) \to v(B).$$

We say that A is *valid in* $\mathfrak{3}$ (in symbols $A \in \mathit{Val}(\mathfrak{3})$) iff $v(A) \in \{0, 1\}$ for every valuation v. We remark that the set $\mathit{Val}(\mathfrak{3})$ is closed under *substitution*, *mp*; and $\mathbf{RMO}^\top_\to \subseteq \mathit{Val}(\mathfrak{3})$.

For any $x \in \{-1, 0, 1\}$, we define $G_x \in \mathit{For}$ as follows:

$$G_{-1} = \top \to (p \to p) \qquad G_0 = p \to p \qquad G_1 = \top$$

PROPOSITION 4.2. *For all* $x, y \in \{-1, 0, 1\}$ *we have:*

$\vdash (G_x \to G_y) \to G_{x \to y}$
$\vdash G_{x \to y} \to (G_x \to G_y)$

PROOF. $\vdash (G_1 \to G_1) \to G_1$ (A5)

$\vdash G_1 \to (G_1 \to G_1)$ (A1)

$\vdash (G_1 \to G_0) \to G_{-1}$ (C1)

$\vdash G_{-1} \to (G_1 \to G_0)$ (C1)

$\vdash (G_0 \to G_1) \to G_1$ (A5)

$\vdash G_1 \to (G_0 \to G_1)$

1. $(p \to p) \to \top$ (A5)
2. $\top \to (\top \to \top)$ (A1)
3. $(p \to p) \to (\top \to \top)$ 1, 2, (A2), *mp*
4. $\top \to ((p \to p) \to \top)$ 3, (A3), *mp*

$\vdash (G_0 \to G_0) \to G_0$ (C4)

$\vdash G_0 \to (G_0 \to G_0)$ (A1)

$\vdash (G_0 \to G_{-1}) \to G_{-1}$ (C4)

$\vdash G_{-1} \to (G_0 \to G_{-1})$

1. $(p \to p) \to ((p \to p) \to (p \to p))$ (A1)
2. $(\top \to (p \to p)) \to (\top \to ((p \to p) \to (p \to p)))$ 1, (C3), *mp*
3. $(\top \to (p \to p)) \to ((p \to p) \to (\top \to (p \to p)))$ 2, (A2), (A3), *mp*

$\vdash (G_{-1} \to G_1) \to G_1$ (A5)

$\vdash G_1 \to (G_{-1} \to G_1)$

1. $\top \to ((\top \to (p \to p)) \to (p \to p))$ (C2)
2. $(p \to p) \to \top$ (A5)
3. $((\top \to (p \to p)) \to (p \to p)) \to ((\top \to (p \to p)) \to \top)$ 2, (A2), (A3), *mp*
4. $\top \to ((\top \to (p \to p)) \to \top)$ 1, 3, (A2), *mp*

$\vdash (G_{-1} \to G_0) \to G_1$ (A5)

$\vdash G_1 \to (G_{-1} \to G_0)$

1. $(\top \to (p \to p)) \to (\top \to (p \to p))$ (C1)
2. $\top \to ((\top \to (p \to p)) \to (p \to p))$ 1, (A3), *mp*

$\vdash (G_{-1} \to G_{-1}) \to G_1$ (A5)

$\vdash G_1 \to (G_{-1} \to G_{-1})$

1. $(\top \to (p \to p)) \to (\top \to (p \to p))$ (C1)
2. $(\top \to (p \to p)) \to (\top \to (\top \to (p \to p)))$ (C7)
3. $(\top \to (p \to p)) \to (\top \to (\top \to (p \to p)))$ 1, 2, (A2), *mp*
4. $\top \to ((\top \to (p \to p)) \to (\top \to (p \to p)))$ 3, (A3), *mp* □

For any valuation v in $\mathfrak{Z}$, we define the corresponding substitution s_v as follows (for any $a \in \mathit{Var}$): $s_v(a) := G_{v(a)}$. Note that $s_v(A)$ results from A by substituting (in a uniform way) $G_{v(a)}$ for every propositional variable a occurring in A.

LEMMA 4.3. *For any $A \in \mathit{For}$, $\vdash s_v(A) \to G_{v(A)}$ and $\vdash G_{v(A)} \to s_v(A)$.*

PROOF. By induction on the complexity of A.

If $A \in \mathit{Var}$, then $s_v(A) = G_{v(A)}$. So the lemma is true by (C1).

Assume that the lemma holds for simpler formulas. If $A = \top$, then $v(\top) = 1$ and $G_{v(A)} = \top$, so $s_v(A) = A = G_{v(A)}$. Thus, we may assume that $A = B \to C$. Let $s = s_v$. By the induction hypothesis, we have: $\vdash s(B) \to G_{v(B)}$, $\vdash G_{v(B)} \to s(B)$, $\vdash s(C) \to G_{v(C)}$ and $\vdash G_{v(C)} \to s(C)$. We only show that $\vdash s(A) \to G_{v(A)}$. By (A2), we have $\vdash (G_{vB} \to sB) \to ((sB \to sC) \to (G_{vB} \to sC))$. By (C3), we have $\vdash (sC \to G_{vC}) \to ((G_{vB} \to sC) \to (G_{vB} \to G_{vC}))$. Hence, by *mp*, $\vdash (sB \to sC) \to (G_{vB} \to sC)$ and $\vdash (G_{vB} \to sC) \to (G_{vB} \to G_{vC})$. So, by (A2) and *mp*, we have $\vdash sA \to (G_{vB} \to G_{vC})$.

Also by Proposition 4.2, $\vdash (G_{v(B)} \to G_{v(C)}) \to G_{v(A)}$. Therefore, by (A2) and *mp*, we have $\vdash s(A) \to G_{v(A)}$, as required. □

By an *extension* of $\mathbf{RMO}^{\top}_{\to}$ we mean a set $\boldsymbol{L} \subseteq \mathit{For}$ containing $\mathbf{RMO}^{\top}_{\to}$ and closed under substitution and modus ponens.

COROLLARY 4.4. *For any extension $\boldsymbol{L}$ of $\mathbf{RMO}^{\top}_{\to}$ we have:*

$$\boldsymbol{L} \text{ has the variable-sharing property iff } (\mathtt{P}) \notin \boldsymbol{L}.$$

PROOF. "⇒" Suppose that $\boldsymbol{L}$ has *VSP* but $(\mathtt{P}) \in \boldsymbol{L}$. Then $q \to (p \to p) \in \boldsymbol{L}$, by (A3) and *mp*. Hence $\boldsymbol{L}$ lacks *VSP*, which is a contradiction.

"⇐" Suppose that $(\mathtt{P}) \notin \boldsymbol{L}$ but $\boldsymbol{L}$ lacks *VSP*. Then some A and B share no variable but $A \to B \in \boldsymbol{L}$. Let v be a valuation in $\mathfrak{Z}$ such that $v(a) = 1$ for every variable a occurring in A, and $v(b) = 0$ for every variable b occurring in B. Then $v(A) = 1$ and $v(B) = 0$. So $v(A \to B) = -1$. Hence $s_v(A \to B) \to (\top \to (p \to p)) \in \boldsymbol{L}$, by Lemma 4.3). So $s_v(A \to B) \to (\mathtt{P}) \in \boldsymbol{L}$, by (A2), (C8), *mp*). Also, $s_v(A \to B) \in \boldsymbol{L}$, because $A \to B \in \boldsymbol{L}$ and $\boldsymbol{L}$ is closed under substitution. So $(\mathtt{P}) \in \boldsymbol{L}$, which is a contradiction. □

We now modify and simplify the concept of a symmetric inference system (introduced in [9]) as follows. The inference rules are fixed (*substitution*, *mp*), so we focus on positive/negative axioms: $\mathbf{S} = (\mathit{POS}, \mathit{NEG})$, where $\mathit{POS} =$ (A1)–(A5) and $\mathit{NEG} = \{\mathsf{P}\} \cup (\mathit{For} - \mathbf{H}_{\to})$.

Let $\boldsymbol{L} \subseteq For$. We say that $\boldsymbol{L}$ is $\mathbf{S}$-*closed* iff $POS \subseteq \boldsymbol{L}$, $NEG \cap \boldsymbol{L} = \emptyset$ and $\boldsymbol{L}$ is closed under substitution and modus ponens. Moreover, we say that a formula A is $\mathbf{S}$-*refutable* iff some $B \in NEG$ is derivable from A by using substitution, *mp* and *POS*.

For any $A \in For$: $A \in Ref(\mathbf{S})$ iff A is $\mathbf{S}$-refutable. Moreover, we put $\mathbf{H}^*_{\rightarrow} := For - Ref(\mathbf{S})$.

PROPOSITION 4.5 (9, Proposition 3.1). *If $\boldsymbol{L}$ is $\mathbf{S}$-closed, then $\boldsymbol{L} \subseteq \mathbf{H}^*_{\rightarrow}$.*

THEOREM 4.6. $\mathbf{H}^*_{\rightarrow} = Val(3) \cap \mathbf{H}_{\rightarrow}$.

PROOF. "$\supseteq$" The set $Val(3) \cap \mathbf{H}_{\rightarrow}$ is closed under substitution and *mp*, because so are $Val(3)$ and $\mathbf{H}_{\rightarrow}$. Also, $Val(3) \cap \mathbf{H}_{\rightarrow}$ contains *POS* and (P) $\notin Val(3)$, so the set $Val(3) \cap \mathbf{H}_{\rightarrow}$ is $\mathbf{S}$-closed. Hence, by Proposition 4.5, $Val(3) \cap \mathbf{H}_{\rightarrow} \subseteq \mathbf{H}^*_{\rightarrow}$.

"$\subseteq$" Assume that $A \notin Val(3) \cap \mathbf{H}_{\rightarrow}$. If $A \notin \mathbf{H}_{\rightarrow}$ then A is $\mathbf{S}$-refutable, so let us assume that $A \notin Val(3)$. Then there is a valuation v in 3 such that $v(A) = -1$. Hence, by Lemma 4.3, we have $\vdash s_v(A) \rightarrow (\top \rightarrow (p \rightarrow p))$. So $\vdash s_v(A) \rightarrow$ (P), by (A2), (C8) and *mp*. Therefore A is $\mathbf{S}$-refutable, So $A \notin \mathbf{H}^*_{\rightarrow}$, which gives the result. □

Remark 4.1. Theorem 4.6 provides the following refutation system axiomatising the complement of $Val(3) \cap \mathbf{H}_{\rightarrow}$ [for more on refutation systems see, e.g., 11]:
Refutation axioms: Every $A \in NEG$.
Refutation rules:
(*Reverse substitution*) B/A where B is a substitution instance of A.
(*Reverse modus ponens* ($\mathbf{RMO}^{\top}_{\rightarrow}$)) B/A where $A \rightarrow B \in \mathbf{RMO}^{\top}_{\rightarrow}$. □

Let $\boldsymbol{L} \subseteq For$ be closed under substitution and modus ponens. We say that $\boldsymbol{L}$ is a *relevant analogue* of $\mathbf{H}_{\rightarrow}$ iff $\mathbf{RMO}^{\top}_{\rightarrow} \subseteq \boldsymbol{L} \subseteq \mathbf{H}_{\rightarrow}$ and $\boldsymbol{L}$ has the variable-sharing property. From Corollary 4.4 we obtain:

PROPOSITION 4.7. *Let $\boldsymbol{L} \subseteq For$ be closed under* substitution *and* modus ponens. *$\boldsymbol{L}$ is a relevant analogue of $\mathbf{H}_{\rightarrow}$ iff $\boldsymbol{L}$ is $\mathbf{S}$-closed.*

COROLLARY 4.8. *$\mathbf{H}^*_{\rightarrow}$ is the greatest relevant analogue of $\mathbf{H}_{\rightarrow}$.*

PROOF. By Theorem 4.6, $\mathbf{H}^*_{\rightarrow}$ is $\mathbf{S}$-closed. Also, by Proposition 4.5, if $\boldsymbol{L}$ is $\mathbf{S}$-closed, then $\boldsymbol{L} \subseteq \mathbf{H}^*_{\rightarrow}$. Hence, by Proposition 4.7, $\mathbf{H}^*_{\rightarrow}$ is the greatest relevant analogue of $\mathbf{H}_{\rightarrow}$. □

COROLLARY 4.9. *$\mathbf{H}^*_{\rightarrow}$ is the greatest extension of $\mathbf{RMO}^{\top}_{\rightarrow}$ that is weaker than $\mathbf{H}_{\rightarrow}$.*

PROOF. Let $\mathbf{RMO}^{\top}_{\rightarrow} \subseteq \boldsymbol{L} \subset \mathbf{H}_{\rightarrow}$. Then (P) $\notin \boldsymbol{L}$. (Otherwise $\mathbf{H}_{\rightarrow} \subseteq \boldsymbol{L}$, so $\mathbf{H}_{\rightarrow} = \boldsymbol{L}$, which is impossible.) So $\boldsymbol{L}$ is a relevant analogue of $\mathbf{H}_{\rightarrow}$ (by Proposition 4.7). By Corollary 4.8, $\boldsymbol{L} \subseteq \mathbf{H}^{*}_{\rightarrow}$. Also, by Theorem 4.6, $\mathbf{RMO}^{\top}_{\rightarrow} \subseteq \mathbf{H}^{*}_{\rightarrow} \subset \mathbf{H}_{\rightarrow}$, which gives the result. □

5. Miscellany

5.1. Extensions of $\mathbf{RMO}^{\top}_{\rightarrow}$

We are going to use the following formulas, where we write $|A|$ for $A \rightarrow A$, for any formula A [see 3]:

$$(|q| \rightarrow |p|) \rightarrow (((p \rightarrow q) \rightarrow p) \rightarrow p) \tag{B1}$$

$$((p \rightarrow |q|) \rightarrow p) \rightarrow p \tag{B2}$$

$$(|p| \rightarrow |r|) \rightarrow ((|q| \rightarrow |r|) \rightarrow (|p \rightarrow q| \rightarrow |r|)) \tag{B3}$$

Firstly, notice that:

PROPOSITION 5.1. $\mathbf{H}^{*}_{\rightarrow} \neq \mathit{Val}(3)$.

PROOF. It is easy to check that (B1) $\in \mathit{Val}(3)$. But (B1) $\notin \mathbf{H}_{\rightarrow}$ because $|q| \rightarrow |p| \in \mathbf{H}_{\rightarrow}$ and (B1) $\mathbf{H}_{\rightarrow}$-entails Peirce's law $((p \rightarrow q) \rightarrow p) \rightarrow p$, which is not in $\mathbf{H}_{\rightarrow}$ [see, e.g., 6]. □

Secondly, we prove that the class of proper extensions of $\mathbf{RMO}^{\top}_{\rightarrow}$ that are weaker than $\mathbf{H}_{\rightarrow}$ has more elements than one.

Let $\mathcal{S} = \langle S, \sqcup, 0\rangle$ be a join semilattice with zero, i.e., for any $x, y \in S$ we have: $0 \sqcup x = x$, $x \sqcup y = y \sqcup x$, $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$, $x \sqcup x = x$. Recall that a *model for* $\mathbf{RMO}^{\top}_{\rightarrow}$ based on $\mathcal{S}$ is any pair $\langle \mathcal{S}, V\rangle$, where V is a function (*valuation*) assigning a subset of S to each propositional variable a such that for all $x, y \in S$: $x \sqcup y \in V(a)$ iff $x, y \in V(a)$ [see, e.g., 2]. We extend V to all formulas as follows (we will write $x \models A$ instead of $x \in V(A)$):

- $x \models A \rightarrow B$ iff for any $y \in S$ either not $y \models A$ or $x \cup y \models B$,
- $x \models \top$.

We say that a formula A is *true in* a model $\langle \mathcal{S}, V\rangle$ iff $0 \models A$. We say that A is *valid in* $\mathcal{S}$ (in symbols $A \in \mathit{Val}(\mathcal{S})$) iff A is true in any model $\langle \mathcal{S}, V\rangle$ based on $\mathcal{S}$. We remark that $\mathit{Val}(\mathcal{S})$ is closed under substitution and *mp*. Moreover, $\mathbf{RMO}^{\top}_{\rightarrow} \subseteq \mathit{Val}(\mathcal{S})$.

For any $n \in \{1, 2, \ldots\}$, let $\mathcal{S}_n$ be the join semilattice consisting of all subsets of the set $\{1, \ldots, n\}$, where $\mathsf{0} := \emptyset$ and $\sqcup$ is the set-theoretic operation $\cup$. We will write 0 for $\emptyset$ and $x_1 \ldots x_i$ for $\{x_1, \ldots, x_i\}$, where $x_1, \ldots, x_i$ belong to $\{1, \ldots, n\}$. Notice that $\mathcal{S}_1$ has two elements 0 and 1 and $\mathcal{S}_3$ has eight elements: 0, 1, 2, 3, 12, 23, 13, 123.

LEMMA 5.2. (i) (B2) $\notin$ *Val*$(\mathcal{S}_1)$.
(ii) (B3) $\notin$ *Val*$(\mathcal{S}_3)$.
(iii) (B3) $\in$ *Val*$(\mathcal{S}_1)$.

PROOF. (i) Let $\models$ be a valuation in $\mathcal{S}_1$ such that $1 \models p$, $1 \not\models q$, $0 \not\models p$ and $0 \models q$. Then $1 \not\models |q|$. So $0 \not\models p \to |q|$. Hence $0 \models (p \to |q|) \to p$, because $1 \models p$. Therefore $0 \not\models$ (B2).

(ii) Let $\models$ be a valuation in $\mathcal{S}_3$ such that $0 \models p$, $0 \models q$, $3 \models p$, $3 \models r$ and moreover:

- if $x \neq 0$ then $x \not\models q$,
- if $x \neq 3$ then $x \not\models r$.
- if $x \notin \{0, 3\}$ then $x \not\models p$.

Then $123 \not\models |r|$ and $123 \models |p \to q|$, because $x \not\models p \to q$ for every x. So $0 \not\models |p \to q| \to |r|$ and if $x \neq 0$, then $x \not\models |q|$. So $0 \models |q| \to |r|$, because $0 \models |r|$). Hence $0 \not\models (|q| \to |r|) \to (|p \to q| \to |r|)$. Moreover, $0 \models |p| \to |r|$ (for both $x \not\models |p|$ if $x \notin \{0, 3\}$ and $3 \models |r|$). Therefore, $0 \not\models$ (B3).

(iii) Suppose that $0 \not\models$ (B3) for some valuation $\models$ in $\mathcal{S}_1$. Then for some y we have $y \models |p| \to |r|$ and $y \not\models (|q| \to |r|) \to (|p \to q| \to |r|)$. We consider the following two cases.

For $y = 1$ we have $1 \models |r|$, because $0 \models |p|$. But for any formulas A, B either $1 \models A \to B$ or $1 \not\models B$. So $1 \not\models |r|$. This is a contradiction.

For $y = 0$ we consider two subcases. First, $1 \models |q| \to |r|$ and $1 \not\models |p \to q| \to |r|$. Hence $1 \models |r|$, since $0 \models |q|$. But $1 \not\models |r|$. This is a contradiction. Second, $0 \models |q| \to |r|$ and $0 \not\models |p \to q| \to |r|$. Since $0 \models |r|$, we get $1 \models |p \to q|$ and $1 \not\models |r|$. Moreover, since we have assumed that $0 \models |p| \to |r|$, we get $1 \not\models |p|$ and $1 \not\models |q|$. Hence $0 \models p$, $1 \not\models p$, $0 \models q$, $1 \not\models q$. So $0 \models p \to q$ and $1 \not\models p \to q$. Therefore $1 \not\models |p \to q|$. This is a contradiction. □

We now establish the following facts:

PROPOSITION 5.3. (I) (B2) $\in \mathbf{H}^*_{\to}$.
(II) $\mathbf{L}_1 \subseteq \mathbf{H}_{\to}$, *where* $\mathbf{L}_1 := Val(\mathcal{S}_1) \cap \mathbf{H}_{\to}$.

(III) $\mathbf{RMO}^{\top}_{\rightarrow} \subseteq \boldsymbol{L}_1$.
(IV) $\boldsymbol{L}_1 \subseteq \mathbf{H}^*_{\rightarrow}$.
(V) $\boldsymbol{L}_1 \subset \mathbf{H}^*_{\rightarrow}$.
(VI) (B3) $\in \boldsymbol{L}_1$.
(VII) $\mathbf{RMO}^{\top}_{\rightarrow} \subset \boldsymbol{L}_1$.
(VIII) $\mathbf{RMO}^{\top}_{\rightarrow} \subset \boldsymbol{L}_1 \subset \mathbf{H}^*_{\rightarrow}$.

PROOF. *Ad* (I): Because (B2) $\in$ *Val*(3) and (B2) $\in \mathbf{H}_{\rightarrow}$.
Ad (III): Because $\mathbf{RMO}^{\top}_{\rightarrow} \subseteq Val(\mathcal{S}_1)$ and $\mathbf{RMO}^{\top}_{\rightarrow} \subseteq \mathbf{H}_{\rightarrow}$.
Ad (IV): By (II), (III) and Corollary 4.9.
Ad (V): By (I) and Lemma 5.2(i).
Ad (VI): By Lemma 5.2(iii) and the fact that (B3) $\in \mathbf{H}_{\rightarrow}$.
Ad (VII): By (VI), Lemma 5.2(ii), since $\mathbf{RMO}^{\top}_{\rightarrow} \subseteq Val(\mathcal{S}_3)$.
Ad (VIII): By (V) and (VII). □

5.2. Characterising *NEG*

The refutation system described in Remark 4.1 may seem unsatisfactory because the set *NEG* (of refutation axioms) is infinite. This set, however, is defined in a constructive way. Indeed, $\mathbf{H}_{\rightarrow}$ is characterized by the class of finite binary trees [see, e.g., 6, 8]. So the complement of $\mathbf{H}_{\rightarrow}$ is recursively enumerable. Hence $\mathbf{H}_{\rightarrow}$ is decidable (because $\mathbf{H}_{\rightarrow}$, being finitely axiomatizable, is recursively enumerable as well). Also, $\mathbf{H}^*_{\rightarrow}$ has a nice semantic characterization (3 plus all finite binary trees). Whether $\mathbf{H}^*_{\rightarrow}$ has an elegant syntactic characterization is an open problem.

5.3. *Val*(3)

However, if we relax our assumptions by requiring that our logic should be a subset of $\mathbf{C}_{\rightarrow}$ (the purely implicational fragment of Classical Logic) rather than $\mathbf{H}_{\rightarrow}$, then an elegant syntactic characterization is possible.

The symbol 2 will stand for the (classical) matrix obtained from 3 by removing 0. So $2 = (\{-1, 1\}, \{1\}, \rightarrow)$, and we have: $x \rightarrow y = -1$ iff $x = 1$ and $y = -1$. We put $\mathbf{C}_{\rightarrow} := Val(2)$. Notice that if v and v' are is valuations in 2 (resp. 3) such that $v(a) = v'(a)$ for each $a \in Var$, then $v(A) = v'(A)$, for each $A \in For$.

Let $\mathbf{S}'$ result from $\mathbf{S}$ by replacing *NEG* with $NEG' = \{(\mathtt{P})\} \cup (For - \mathbf{C}_{\rightarrow})$. We define $\mathbf{C}^*_{\rightarrow}$ to be the set of all formulas that are not **S'**-refutable. Notice that, respectively in virtue of the proof of Theorem 4.6 and the fact that $Val(3) \subseteq \mathbf{C}_{\rightarrow}$, we obtain:

- $\mathit{Val}(3) \cap \mathbf{C}_{\rightarrow} = \mathbf{C}^{*}_{\rightarrow}$,
- $\mathit{Val}(3) = \mathbf{C}^{*}_{\rightarrow}$.

5.3.1. Refutation system for *Val*(3)

Since $\mathit{Val}(3) \subseteq \mathbf{C}_{\rightarrow}$, it can be shown (by using Lemma 4.3 and the fact that $\mathit{Val}(3) = \mathbf{C}^{*}_{\rightarrow}$; see the proof of Theorem 4.6) that the following refutation system axiomatizes the complement of $\mathit{Val}(3)$:

- refutation axiom: (P);
- refutation rules: reverse substitution, reverse modus ponens ($\mathbf{RMO}^{\top}_{\rightarrow}$).

We remark that our axiomatization is simpler than any (positive) axiomatization for $\mathit{Val}(3)$ that can be found in the literature [see 3].

5.4. Open problems

Can we obtain such results without $\top$ (or without the mingle axiom)? It seems hard. Anyway, we leave the following open problems:

1. Let $\mathbf{S}_1 = (\mathit{POS}_1, \mathit{NEG}_0)$, where POS_1 is the set of the $\mathbf{RMO}_{\rightarrow}$ axioms and $\mathit{NEG}_0 := \{A \rightarrow B : A, B \text{ share no variable}\}$. Characterize $\mathit{Ref}(\mathbf{S}_1)$ or its complement.
2. Let $\mathbf{S}_2 = (\mathit{POS}_2, \mathit{NEG}_0)$, where POS_2 is the set of the $\mathbf{R}_{\rightarrow}$ axioms. Characterize $\mathit{Ref}(\mathbf{S}_2)$.

6. A correction to [10]

Recall that we are now dealing with the set FOR of all formulas generated from Var by $\neg$, $\wedge$, $\vee$ and $\rightarrow$. In the proof of Lemma 2 in [10, p. 69] the inference from $P \rightarrow (s_v(C) \equiv G_{v(C)}) \in \mathbf{RM}$ and $P \rightarrow (s_v(D) \equiv G_{v(D)}) \in \mathbf{RM}$ to $P \rightarrow (\neg s_v(C) \equiv \neg G_{v(C)}) \in \mathbf{RM}$ and $P \rightarrow ((s_v(C) \otimes c_v(D)) \equiv (G_{v(C)} \otimes G_{v(D)})) \in \mathbf{RM}$, where $\otimes \in \{\wedge, \vee, \rightarrow\}$, is justified by modus ponens and

$$(3) \qquad (A \rightarrow (B \equiv C)) \rightarrow (A \rightarrow (D \equiv D(B/C))),$$

where $D(B/C)$ results from D by replacing some occurrences of B by C. But neither (3) nor the preceding formula belongs to $\mathbf{RM}$.

Indeed, let $F := (r \rightarrow (p \equiv p)) \rightarrow (r \rightarrow (p \wedge q \equiv p \wedge q))$ and $G := (p \equiv p) \rightarrow (p \wedge q \equiv p \wedge q)$, and let v be a valuation in 3 such that $v(p) = 1$, $v(q) = 0$, $v(r) = 1$. Then $v(F) = -1$ and $v(G) = -1$, so these formulas are not $\mathbf{RM}$ laws.

However, the above inference is correct, but (3) should be replaced with

(3′) if $P \to (A \equiv B) \in \mathbf{RM}$ then $P \to (H \equiv H(A/B)) \in \mathbf{RM}$,

where $P = p \wedge \neg p$. We now outline a proof of (3′).

LEMMA 6.1. *The following formulas are in* **RM**.

$$A \wedge B \to A \qquad A \wedge B \to B$$
$$A \wedge B \to B \wedge A$$
$$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$$
$$(A \to B) \to ((C \to A) \to (C \to B))$$
$$(A \to B) \to (\neg B \to \neg A)$$
$$(A \to B) \wedge (C \to C) \to ((A \wedge C) \to (B \wedge C)) \qquad (\star)$$
$$(A \to B) \wedge (C \to C) \to ((A \vee C) \to (B \vee C))$$

PROOF. We only check ($\star$). Let v be a valuation in $\mathfrak{M}$. We consider three cases.

1. $v(C) \geq \max(v(A), v(B))$. Then $v(A \wedge C) = v(A)$ and $v(B \wedge C) = v(B)$. So $v(A \wedge C \to B \wedge C) = v(A \to B)$.

2. $v(C) \leq \min(v(A), v(B))$. Then $v(A \wedge C) = v(C)$ and $v(B \wedge C) = v(C)$. So $v(A \wedge C \to B \wedge C) = v(C \to C)$.

3. $\min(v(A), v(B)) < v(C) < \max(v(A), v(B))$. If $v(A) \leq v(B)$, then $v(A \wedge C) = v(A)$ and $v(B \wedge C) = v(C)$. Moreover. if $v(C) \geq 0$, then $v(C \to C) = C$, and so $v((A \to B) \wedge (C \to C)) \leq v(C) \leq v(A \to C) = v((A \wedge C) \to (B \wedge C))$. If, however, $v(C) < 0$, then $v(C \to C) = v(\neg C)$. Also, $v(\neg C) \leq v(\neg A)$, because $v(A) \leq v(C)$. Hence $v((A \to B) \wedge (C \to C)) \leq v(\neg C) \leq v(\neg A) \leq v(A \to C) = v((A \wedge C) \to (B \wedge C))$.

If $v(A) > v(B)$, then $v(A \wedge C) = v(C)$ and $v(B \wedge C) = v(B)$. Also, $v(\neg A) \leq v(\neg C)$, so $\min(v(\neg A), v(B)) \leq v(\neg C)$. Hence $v((A \to B) \wedge (C \to C)) \leq \min(v(\neg A), v(B)) \leq \min(v(\neg C), v(B)) = v((A \wedge C) \to (B \wedge C))$. □

LEMMA 6.2. *For any* $A \in \mathbf{RM}$, $P \to A \in \mathbf{RM}$.

PROOF. Assume that $A \in \mathbf{RM}$ and v be a valuation in $\mathfrak{M}$. Then $v(P) \leq 0$ and $v(A) \geq 0$. Hence $v(P \to A) \in \mathcal{D}$. Therefore, $P \to A \in \mathbf{RM}$. □

PROPOSITION 6.3. *If* $P \to (A \equiv B) \in \mathbf{RM}$ *then* $P \to (H \equiv H(A/B)) \in \mathbf{RM}$.

Proof. By induction on the complexity of H.

If $H \in Var$, then $H = H(A/B)$, so $P \to (H \equiv H(A/B)) \in \mathbf{RM}$, by Lemma 6.2.

Suppose that the proposition holds for simpler formulas. We only check the case where $H = C \wedge D$. Assume that $P \to (A \equiv B) \in \mathbf{RM}$. By the induction hypothesis, we have:

- if $P \to (A \equiv B) \in \mathbf{RM}$ then $P \to (C \equiv C(A/B)) \in \mathbf{RM}$,
- if $P \to (A \equiv B) \in \mathbf{RM}$ then $P \to (D \equiv D(A/B)) \in \mathbf{RM}$.

Moreover, $P \to (D \to D) \in \mathbf{RM}$, by Lemma 6.2. Hence, by Lemma 6.1, *mp* and *adjunction*) we get:

- $P \to (C \equiv C(A/B)) \wedge (D \to D) \in \mathbf{RM}$.

Since $(C \equiv C(A/B)) \wedge (D \to D) \to (C \to C(A/B)) \wedge (D \to D) \in \mathbf{RM}$ and $(C \to C(A/B)) \wedge (D \to D) \to (C \wedge D \to C(A/B) \wedge D) \in \mathbf{RM}$, we finally obtain $P \to (C \wedge D \equiv C(A/B) \wedge D) \in \mathbf{RM}$, by Lemma 6.1, *mp*, *adjunction*.

In a similar way, the following is established:

- $P \to (C(A/B) \wedge D \equiv C(A/B) \wedge D(A/B)) \in \mathbf{RM}$.

Therefore, by (2) in [10] and *mp*, $P \to (H \equiv H(A/B)) \in \mathbf{RM}$. □

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