REFUTATION SYSTEMS

Tomasz Skura

Abstract

I present some of my contributions to the topic. For more concepts/methods/results/applications/questions on the subject, see the programme and the slides of the talks presented during Refutation Symposium 2018.

In the literature, there are two approaches to refutation: an indirect one and a direct one.

In the indirect approach, you refute a formula by failing to prove it. For example, you search for a proof of A, and if all the possibilities of finding a proof for A have been exhausted, you say that A is refuted.

In the direct approach, a single refutation of A, which is a derivation, justifies refuting A.

Although the indirect approach is standard, we believe that combining both approaches is more attractive and can yield new results that are both interesting and useful. Note that if A is non-valid, then finding a refutation for A may be simpler than producing all possibilities of proving it.

A refutation system is just like a traditional axiomatic system, but it is applied to non-valid formulas rather than valid ones.

It consists of refutation axioms and refutation rules. We say that a formula A is refutable (in symbols $\dashv A$) iff it is derivable from refutation axioms by refutation rules. There are various kinds of refutation rules. Let us start with the following refutation rules introduced by Łukasiewicz [JL 1951].

(*reverse substitution* (or RS)) B/A where B is a substitution instance of A.

(*reverse modus ponens* (or RMP)) B/A where $\vdash A \to B$.

Systems with RS

Let **L** be a logic, that is, a set of formulas closed under *substitution*, *modus ponens*, and possibly some other rules (e.g. *necessitation*).

We say that a refutation system is characteristic for **L** iff we for every formula A, we have:

$A \notin \mathbf{L}$ iff $\dashv A$.

For example, **CL** (Classical Propositional Logic) can be characterized by the following simple refutation system.

Refutation axiom: $\bot$ (the false).
Refutation rules: RS, RMP.

Refutation systems of this kind (having certain characteristic formulas of finite algebras as refutation axioms) characterize every intermediate logic (and every normal modal logic) with the FMP (finite model property); however, there are logics without the FMP that do have finite refutation systems (see [TS 1992, 1994, 2013]).

Furthermore, there are logics with problematic (or non-existent) proof theories, but having neat syntactic descriptions of their non-validities. For example, Medvedev's logic is characterized by the following refutation system (see [TS 1992]).

Refutation axiom: $\bot$ (the false).
Refutation rules: RS, RMP_{KP}, RD, where

(RMP_{KP}) B/A where $\vdash_{KP} A \to B$.
(Here $\vdash_{KP}$ means provability in the Kreisel-Putnam logic.)

(RD) $A, B/A \vee B$ (This rule reverses the disjunction property.)

Also, refutation systems with RS are useful for establishing certain facts about the lattice of extensions of a given logic, especially, concerning maximality and minimality (see [TS 2004, 2009, 2017a]).

Thus, the main applications of systems with RS are:

1. Specific/generic descriptions of the non-validities of logics.
2. Establishing maximality/minimality in the lattices of logics.

Systems without RS

Constructing Counter-Models

Of course, *reverse substitution* is not good for constructing counter-models. However, *reverse modus ponens* is OK, but in Johansson's logic and extensions (including **Int** and intermediate logics), it must have the following form.

(RMP') $A \to C/B \to C$ where $\vdash A \to B$.

Roughly speaking, in Johansson's logic and extensions as well as in normal modal logics, finite countermodels can be constructed from syntactic refutations (which can be presented as finite trees consisting of formulas) as follows.

- For every formula A, we construct its Mints-normal form F_A such that $\vdash A$ iff $\vdash F_A$ (see [GM 1990]). Every normal form has its natural number called its rank (see [TS 2011, 2013]).
- We give a Scott-style refutation rule involving normal forms (see [DS 1957]):

  $(R) \qquad \dfrac{F_1, ..., F_k}{F}$

  where F is a normal form of rank $k > 0$ and each F_i is (after simple *modus ponens* transformations) a normal form of rank $< k$.
- Our refutation system consits of refutation axioms (which are normal forms of rank 0 that are non-valid) and refutation rules: R and RMP (or RMP').
- We prove, by a simple inductive argument, that every normal form F is either provable or refutable (see [DS 1957], [TS 2011, 2013]). So, if F is not provable then F has a refutation tree.
- We transform every syntactic refutation tree into a Kripke frame by removing the nodes obtained by RMP and by defining a suitable accessibility relation (see [TS 2002, 2013, 2017a]). From the normal forms, we extract a valuation falsifying the refutable formulas. So, if F is a node in a syntactic refutation tree, then it is false at some point in the

corresponding model built from this tree. (We remark that the corresponding frames need not be trees.) Hence if F is refutable then F has a countermodel, so F is not provable.

- As a result we get both syntactic completeness (F is not provable iff F is refutable) and semantic completeness (F is provable iff F is valid in all finite tree-type frames).

We remark that a similar method for sequents (rater than Mints-style normal forms) was given by Pinto and Dyckhoff [P-D 1995].

Decision Procedures

Of course, RMP is not good for refutation search procedures. However, we do not need the whole RMP in our syntactic completeness proof (see [TS 2011]). Just a few simple auxiliary rules are enough. The completeness proof provides a refutation search procedure that is a finite tree consisting of finite sets of formulas and having the following property (see [TS 2017b]).

The origin is non-valid iff some end node is non-valid.
(Here we say that a set of formulas is non-valid iff every member of it is non-valid.) Note that it is in fact a decision procedure.

Thus, the main applications of systems without RS are:
1. Constructive completeness proofs that are simple.
2. Refined semantic characterizations of logics by finite tree-type frames.
3. Loop-free constructions of counter-models.
4. Refutation search procedures that, at least in some cases, are simpler than those provided by standard methods (see [TS 2013] p. 125).

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