Notre Dame Journal of Formal Logic
Volume XIX, Number 1, January 1978
NDJFAM

A FORMAL SYSTEM FOR THE NON-THEOREMS OF THE PROPOSITIONAL CALCULUS

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Introduction The completeness of the classical propositional calculus allows us to give a deductive system consisting of finitely many *axiom schemas* and finitely many *rules of inference*, that permit us to pass from a formula or a pair of formulae to a syntactically related formula, in such a manner that the formulae obtained inductively from the axioms by repeated application of the rules are exactly the tautologies. In this paper we give an analogous deductive system (more concretely, a Hilbert type system) such that the formulae deduced are exactly those that *are not* tautologies, the non-theorems of the propositional calculus. Obviously, this has to be the most non-standard of the non-classical logics. It is important to note that there are many other algorithms to generate recursively the non-theorems, since the propositional calculus is decidable. Usually they are based in the methodical search for a counterexample, but they lack the inductive character of a Hilbert type system, where every formula involved in a deduction is itself deductible. In our system, unlike semantic tableaux or refutation trees, every formula introduced in a deduction is a non-tautology, and it is introduced only if it is a non-tautological axiom, or it follows by one of the non-tautological rules of inference from non-tautologies introduced earlier in the deduction.

1 *Axioms and rules* We assume that the only connectives are $\sim$ and $\supset$. $p, q, p_1, p_2, \ldots$ denote atomic formulae. $\alpha, \beta, \gamma, \ldots$ denote arbitrary formulae. We define $\mathcal{P}(\alpha) = \{p \mid p \text{ occurs in } \alpha\}$.

Axioms

A1 $p \supset \sim p$ (p atomic)
A2 $\sim p \supset p$ (p atomic)

Rules

R1 (a) $\dfrac{\alpha}{p \supset \alpha}$ (p atomic, p does not occur in α)

Received May 23, 1977

R1 (b) $\dfrac{\alpha}{\sim p \supset \alpha}$ (p atomic, p does not occur in α)

R2 $\dfrac{\alpha \supset \beta}{\alpha \supset (\alpha \supset \beta)}$

R3 $\dfrac{\alpha \supset \beta}{(\gamma \supset \alpha) \supset \beta}$

R4 $\dfrac{\sim\alpha \supset \beta}{(\alpha \supset \gamma) \supset \beta}$

R5 $\dfrac{\sim\alpha \supset \beta}{\alpha}$

R6 $\dfrac{\alpha \supset \beta}{\sim\sim\alpha \supset \beta}$

R7 $\dfrac{\alpha \supset (\beta \supset \gamma)}{\beta \supset (\alpha \supset \gamma)}$

R8 $\dfrac{\alpha \supset S,\ \sim\beta \supset S}{\sim(\alpha \supset \beta) \supset S}$ (where S has the form indicated below)

The formula S in R8 must have the form $S = S_1$ or $S = S_1 \supset (S_2 \supset \dots (S_{n-1} \supset S_n) \dots)$, with $S_i = p_i$ or $S_i = \sim p_i$, $p_i \neq p_j$ for $i \neq j$, and $\mathcal{P}(\alpha \supset \beta) \subseteq \{p_1, p_2, \dots, p_n\}$.

Note that the axioms cannot be replaced with schemata, and a substitution rule cannot be allowed, since many non-tautologies become tautologies through substitution. We use the notation $\Vdash\alpha$ to indicate that the formula α is deducible in the above system.

Examples

1. $\Vdash(p \supset q) \supset (q \supset p)$	1. $\sim p \supset p$	A2
	2. p	R5
	3. $q \supset p$	R1
	4. $q \supset (q \supset p)$	R2
	5. $(p \supset q) \supset (q \supset p)$	R3
2. $\Vdash\sim p$	1. $p \supset \sim p$	A1
	2. $\sim\sim p \supset \sim p$	R6
	3. $\sim p$	R5
3. $\Vdash\sim(p \supset p)$	1. $p \supset \sim p$	A1
	2. $(p \supset p) \supset \sim p$	R3
	3. $\sim\sim(p \supset p) \supset \sim p$	R6
	4. $\sim(p \supset p)$	R5

4. $\Vdash((p \supset \sim p) \supset (\sim q \supset q)) \supset q$

We give the "proof" in tree form, since in this example the use of R8 seems essential:

$$
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\begin{array}{ll}
\text{A2} & \sim q \supset q \\
\text{R5} & q \\
\text{R1} & p \supset q \\
\text{R2} & p \supset (p \supset q)
\end{array}
\qquad
\begin{array}{ll}
\text{A2} & \sim q \supset q \\
\text{R5} & q \\
\text{R1} & p \supset q \\
\text{R2} & p \supset (p \supset q) \\
\text{R6} & \sim\sim p \supset (p \supset q)
\end{array}
}{
\text{R8} \quad \sim(p \supset \sim p) \supset (p \supset q)
}
}{
\text{R4} \quad ((p \supset \sim p) \supset (\sim q \supset q)) \supset (p \supset q)
}
\qquad
\begin{array}{ll}
\text{A2} & \sim q \supset q \\
\text{R1} & p \supset (\sim q \supset q) \\
\text{R7} & \sim q \supset (p \supset q)
\end{array}
}{
\begin{array}{ll}
\text{R8} & \sim(((p \supset \sim p) \supset (\sim q \supset q)) \supset q) \supset (p \supset q) \\
\text{R5} & ((p \supset \sim p) \supset (\sim q \supset q)) \supset q
\end{array}
}
$$

2 *Completeness* As usual, $\models \alpha$ means that α is a tautology. We show that our system is perfectly *unsound* and completely *antitautological*. In other words, we prove the following

Theorem A. *If* $\dashv\vdash \alpha$ *then not* $\models \alpha$.
B. *If not* $\models \alpha$ *then* $\dashv\vdash \alpha$.

Proof: A. We use the symbol $\not\models \alpha$ to indicate that there is a valuation v such that $\mathsf{v}(\alpha) = \mathbf{F}$. It is clear that $\not\models p \supset \sim p$ and $\not\models \sim p \supset p$, for p atomic, and rules R1 to R7 preserve this property; in fact, R2, R6, and R7 are logical equivalences and preserve "everything". The only non-trivial case is that of rule R8. Let S be as explained in the rule, and let v and w be valuations such that $\mathsf{v}(\alpha \supset S) = \mathbf{F}$ and $\mathsf{w}(\sim\beta \supset S) = \mathbf{F}$. Then $\mathsf{v}(S) = \mathsf{w}(S) = \mathbf{F}$ and so: $\mathsf{v}(S_i) = \mathsf{w}(S_i) = \mathbf{T}$ for $i < n$, $\mathsf{v}(S_n) = \mathsf{w}(S_n) = \mathbf{F}$. But these conditions determine completely the valuations in $p_1, p_2, \ldots, p_n$, thus $\mathsf{v}\restriction\{p_1, p_2, \ldots, p_n\} = \mathsf{w}\restriction\{p_1, p_2, \ldots, p_n\} = \mathsf{v}^*$. Since $\mathcal{P}(\alpha) \cup \mathcal{P}(\beta) \subseteq \{p_1, p_2, \ldots, p_n\}$, we have $\mathsf{v}^*(\alpha) = \mathsf{v}(\alpha) = \mathbf{T}$, $\mathsf{v}^*(\beta) = \mathsf{w}(\beta) = \mathbf{F}$, $\mathsf{v}^*(S) = \mathsf{v}(S) = \mathsf{w}(S) = \mathbf{F}$, and so $\mathsf{v}^*(\sim(\alpha \supset \beta) \supset S) = \mathbf{F}$. This finish the proof.

B. We prove first, by induction in the complexity of the formula α, the following property:

$$
(*) \quad \begin{cases} \text{If } \mathcal{P}(\alpha) \subseteq \{p_1, p_2, \ldots, p_n\},\ S = S_1 \supset (S_2 \supset \ldots (S_{n-1} \supset S_n) \ldots) \text{ with} \\ p_i \neq p_j \text{ for } i \neq j, \text{ and } S_i = p_i \text{ or } S_i = \sim p_i, \text{ then: } \not\models \alpha \supset S \text{ implies } \dashv\vdash \alpha \supset S. \end{cases}
$$

Case I: $\alpha = p_j$ (atomic). Since $\mathsf{v}(p_j \supset S) = \mathbf{F}$, then $\mathsf{v}(p_j) = \mathbf{T}$, $\mathsf{v}(S_i) = \mathbf{T}$ for $i < n$, and $\mathsf{v}(S_n) = \mathbf{F}$.

Subcase I-a: $j < n$. Then $\mathsf{v}(S_j) = \mathsf{v}(p_j) = \mathbf{T}$, this forces $S_j = p_j$ and $S = S_1 \supset (S_2 \supset \ldots (p_j \supset \ldots \supset S_n) \ldots)$. We have the following derivation of $p_j \supset S$:

$$
\begin{array}{ll}
 & S_n \qquad\qquad \text{(as in examples 1 and 2)} \\
\text{R1} & p_j \supset S_n \\
\text{R1} & S_{n-1} \supset (p_j \supset S_n) \\
\text{R7} & p_j \supset (S_{n-1} \supset S_n) \\
\text{(R1 \& R7)} & \vdots
\end{array}
$$

$p_j \supset (S_{j+1} \supset \ldots (S_{n-1} \supset S_n) \ldots)$
R2 $p_j \supset (p_j \supset (S_{j+1} \supset \ldots (S_{n-1} \supset S_n) \ldots)$
R1 $S_{j-1} \supset (p_j \supset (p_j \supset (S_{j+1} \supset \ldots (S_{n-1} \supset S_n) \ldots)$
R7 $p_j \supset (S_{j-1} \supset (p_j \supset (S_{j+1} \supset \ldots (S_{n-1} \supset S_n) \ldots)$

(R1 & R7) $\vdots$

$p_j \supset (S_1 \supset \ldots (S_{j-1} \supset (p_j \supset (S_{j+1} \supset \ldots (S_{n-1} \supset S_n) \ldots)$
$\Vdash p_j \supset S$

Subcase I-b: $j = n$. Then $v(S_n) = \mathbf{F}$. Since $v(p_n) = v(p_j) = \mathbf{T}$ by the initial observation for Case I, we must have $S_n = \sim p_n$. We have the deduction:

A1 $p_n \supset \sim p_n$
R1 $S_{n-1} \supset (p_n \supset \sim p_n)$
R7 $p_n \supset (S_{n-1} \supset \sim p_n)$

(R1 & R7) $\vdots$

$p_n \supset (S_1 \supset \ldots (S_{n-1} \supset \sim p_n) \ldots)$
$\Vdash p_n \supset S$

Case II: (inductive step) $\alpha = \sim\beta$.

Subcase II-a: $\beta = p_j$ with p_j atomic. It is similar to Case I.

Subcase II-b: $\beta = \sim\gamma$. If $v(\sim\sim\gamma \supset S) = \mathbf{F}$ then $v(\gamma \supset S) = \mathbf{F}$. By induction hypothesis: $\Vdash \gamma \supset S$, by R6: $\Vdash \sim\sim\gamma \supset S$.

Subcase II-c: $\beta = (\gamma \supset \gamma')$. If $v(\sim(\gamma \supset \gamma') \supset S) = \mathbf{F}$ then $v(\gamma) = \mathbf{T}$, $v(\gamma') = \mathbf{F}$, and $v(S) = \mathbf{F}$. Therefore, $v(\gamma \supset S) = \mathbf{F}$ and $v(\sim\gamma' \supset S) = \mathbf{F}$. By induction hypothesis: $\Vdash \gamma \supset S$ and $\Vdash \sim\gamma' \supset S$. By R8: $\Vdash \sim(\gamma \supset \gamma') \supset S$.

Case III: (inductive step) $\alpha = (\gamma \supset \gamma')$. If $v((\gamma \supset \gamma') \supset S) = \mathbf{F}$ then $v(S) = \mathbf{F}$, and $v(\gamma) = \mathbf{F}$ or $v(\gamma') = \mathbf{T}$. In the first case, $v(\sim\gamma \supset S) = \mathbf{F}$. By inductive hypothesis: $\Vdash \sim\gamma \supset S$, and by R4: $\Vdash (\gamma \supset \gamma') \supset S$. In the second case, $v(\gamma' \supset S) = \mathbf{F}$. By inductive hypothesis: $\Vdash \gamma' \supset S$, and by R3: $\Vdash (\gamma \supset \gamma') \supset S$.

To conclude the proof of the theorem, let $v(\alpha) = \mathbf{F}$, $\mathcal{P}(\alpha) = \{p_1, p_2, \ldots, p_n\}$ and define $p_i^v = p_i$ if $v(p_i) = \mathbf{T}$, $p_i^v = \sim p_i$ if $v(p_i) = \mathbf{F}$. Then $v(p_i^v) = \mathbf{T}$ for $i = 1, 2, \ldots, n$. Form the formula $S = p_1^v \supset (p_2^v \supset \ldots (p_{n-1}^v \supset \sim p_n^v) \ldots)$. We have $v(S) = \mathbf{F}$ and $v(\sim\alpha \supset S) = \mathbf{F}$. By property (*) above: $\Vdash \sim\alpha \supset S$, and by R5: $\Vdash \alpha$ Q.E.D.

3 *Observations* If the propositional language contains the connective $\vee$, it is enough to add the following rules to obtain completeness:

R9 (a) $\dfrac{\alpha \supset \beta}{(\alpha \vee \gamma) \supset \beta}$

R9 (b) $\dfrac{\alpha \supset \beta}{(\gamma \vee \alpha) \supset \beta}$

If the system contains the connective $\wedge$, the following rule will be enough to take care of it:

$$\text{R10} \quad \frac{\alpha \supset (\beta \supset \gamma)}{(\alpha \wedge \beta) \supset \gamma}$$

Finally, it is not possible to give a similar deductive system for the non-valid formulae of the first-order predicate calculus because that would imply the decidability of the calculus.

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