Rafał Dutkiewicz

The Method of Axiomatic Rejection for the Intuitionistic Propositional Logic*

To the memory of Jerzy Słupecki

Abstract. We prove that the intuitionistic sentential calculus is Ł-decidable (decidable in the sense of Łukasiewicz), i.e. the sets of theses of Int and of rejected formulas are disjoint and their union is equal to all formulas. A formula is rejected iff it is a sentential variable or is obtained from other formulas by means of three rejection rules. One of the rules is original, the remaining two are Łukasiewicz's rejection rules: by detachement and by substitution. We extensively use the method of Beth's semantic tableaux.

1. Introduction

The method of rejection was introduced into the realm of logic by Jan Łukasiewicz. It was when Łukasiewicz refuted false syllogistic moods using for that purpose an axiomatic rejection method: some false expressions were assumed to be rejected axiomatically, and then, other false expressions were deduced from these axioms by means of some suitable inference rules .

The original rules used by Łukasiewicz, later in the literature called Łukasiewicz's rejection rules, can be formulated as follows:

1. the rule of rejection by detachement: if a formula $X \to Y$ is accepted in the system under consideration and Y is rejected in this system, then X is rejected too,
2. the rule of rejection by substitution: if a substitution instance of the formula X is rejected, so is X.

Moreover, we owe to Łukasiewicz the following theorem: The foregoing two rejection rules and the sole rejected axiom stating that the atomic formula p is rejected suffice for rejecting all non-theses of the classical propositional logic.

One may wonder what was a source of inspiration for Łukasiewicz to deal with the method of rejection. It seems that he was interested in a general question how to refute false expressions (i.e. non-theses of a system) essentially within the system itself, without any semantic devices, and without augmenting

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the system too much, for instance by quantifiers binding propositional variables.

The rejection of false expressions of a formal system is closely related to the so called *Ł*-decidability of the system. Recall that a system *S* is *Ł*-decidable if the set of theses of *S* and the set of rejected formulas of *S* are disjoint and their set-theoretical union is the set of all formulas of *S*. Łukasiewicz dealt with *Ł*-decidability in some of his papers, eg. [3], [4], whereas the very term "*Ł*-decidability" was introduced by Jerzy Słupecki in [13].

Słupecki claimed that the general notion of *Ł*-decidability is exactly what Łukasiewicz had understood by decidability. Słupecki remarked that under suitable assumptions *Ł*-decidability implies decidability: It is enough, for example, to assume that

(a) the sets of all formulas, of all axioms and of all rejected axioms are calculable,
(b) detachement and substitution are the only inference rules,
(c) rejected rules are finitary and the total number of them is finite.

It is worth noting that Słupecki and his disciples from Opole investigated the rejection method deeply and widely. The article "Theory of rejected propositions", [13], summarizes to a great extent the research carried out in the Opole School. It contains also valuable historical comments.

Reverting to Łukasiewicz, let us recall that he conjectured in [6] that the intuitionistic propositional calculus is *Ł*-decidable, and moreover, he thought that the unique axiom stating that a propositional variable is rejected together with the rejection rules by detachement and substitution plus one new rule according to which $A \vee B$ is rejected whenever so are A and B suffice for deducing all non-theses of the intuitionistic calculus.

Today, of course, we know that the new rule proposed by Łukasiewicz is sound but does not suffice for rejecting all non-theses of the intuitionistic calculus. Namely, one can extend the intuitionistic propositional calculus, e.g. by means of the Kreisel-Putnam formula, so that the resulting system is closed under the rule proposed by Łukasiewicz. And what is more, we know that there is infinitely many such extensions (c.f. [9]), and there is no finite set of rejected axioms which together with Łukasiewicz's rules are complete for deriving all non-theses of the intuitionistic calculus [8].

In the light of these negative results there emerge two possibilities in dealing with the method of rejection for intuitionistic logic. Either we look for an infinite set of rejected axioms or we augment the two original Łukasiewicz's rules by new ones. In the present paper we have choosen the latter possibility. It is shown by means of tableau method that if we add one suitable rule to the two original Łukasiewicz's rejection rules and take one rejected axiom according to which a propositional variable is rejected, then the intuitionistic propositional logic is *Ł*-decidable.

After the present work had been completed I acquainted myself with D. Scott's beautiful paper "Completeness proofs for the intuitionistic sentential calculus" [11], in which, among other interesting things, there is a proof of Ⱡ-decidability of the intuitionistic propositional logic. One must admit that the Scott's paper has been unknown to Słupecki and the logicians of his circle. In spite of Scott's priority I have decided to publish the present paper because my rule of rejection and my method of proof of Ⱡ-decidability are quite different and independent of Scott's.

2. The method of semantic tableaux for the intuitionistic propositional calculus

2.1. The system *FI*. Let W be the set of all formulas of the intuitionistic propositional calculus (INT, for short) including the propositional constant 0 (falsum) and 1 (verum).

Let Γ and Q be arbitrary finite subsets of W. Then the pair $\langle \Gamma, Q\rangle$ will be called a *tableau* and denoted by

(1) $T(\Gamma|Q)$.

We say that the formulas belonging to Γ (resp. to Q) form the left (resp. the right) column of the tableau. If $\Gamma = \{A_1, \ldots, A_n\}$, then tableau (1) can be depicted as $T(A_1, \ldots, A_n|Q)$. We also write $T(\Gamma_1, \Gamma_2|Q)$ for $T(\Gamma_1 \cup \Gamma_2|Q)$ and $T(\Gamma|Q_1, Q_2)$ instead of $T(\Gamma|Q_1 \cup Q_2)$.

The capital letter T, possibly with indices, will serve as a variable denoting arbitrary tableaux.

A *tableau transforming rule* is to be defined as a pair $\langle T_1, T_2\rangle$ or a triple $\langle T_1, T_2, T_3\rangle$ of tableaux. The rules will be depicted in the form of the following schemata:

(2) $\mathrm{r}\,\dfrac{T_1}{T_2}$ (3) $\mathrm{r_a}\,\dfrac{T_1}{T_2,\ T_3}.$

We say of the tableau T_2, and of the tableaux T_2, T_3, that they are *derived* from the tableau T_1 by the rule r or by the rule $\mathrm{r_a}$ respectively. This will be alternatively written down as

(4) $T_1\,\mathrm{r}\,T_2$ and (5) $T_1\mathrm{r_a}T_2,\quad T_1\mathrm{r_a}T_3$.

Any rule of schema (3) is called a *branch* rule. We say that this rule causes alternative branchings.

We take under consideration the following set *FI* of tableau transforming rules. They are essentially the same as Fitting's ones [2].

$$\text{rK}_1 \quad \frac{T(\Gamma, A \wedge B|Q)}{T(\Gamma, A, B|Q)} \qquad \text{rK}_2 \quad \frac{T(\Gamma|Q, A \wedge B)}{T(\Gamma|Q, A) \quad T(\Gamma|Q, B)}$$

$$\text{rA}_1 \quad \frac{T(\Gamma, A \vee B|Q)}{T(\Gamma, A|Q) \quad T(\Gamma, B|Q)} \qquad \text{rA}_2 \quad \frac{T(\Gamma|Q, A \vee B)}{T(\Gamma|Q, A, B)}$$

$$\text{rC}_1 \quad \frac{T(\Gamma, A \to B|Q)}{T(\Gamma, A \to B|Q, A) \quad T(\Gamma, B|Q)} \qquad \text{rC}_2 \quad \frac{T(\Gamma|Q, A \to B)}{T(\Gamma, A|B)}$$

$$\text{rN}_1 \quad \frac{T(\Gamma, \sim A|Q)}{T(\Gamma, \sim A|A, Q)} \qquad \text{rN}_2 \quad \frac{T(\Gamma|Q, \sim A)}{T(\Gamma, A|\varnothing)}.$$

Tableau proofs have the form of trees. If we want to know whether there is a tableau proof of a formula X, we transform the tableau $T(\varnothing|X)$ by means of the foregoing rules FI. The proof is written down as a tree with $T(\varnothing|X)$ in its origin. The subsequent tableaux are placed below the origin. The constructed tree branches out if one applies a branch rule. The resulted tree is a proof for X provided its every branch ends with a closed tableau, i.e. with a tableau having the same formula in the left and the right column. Here are formal definitions of these explanations:

Let $\mathscr{G} = \langle G, R, T_0\rangle$ be a triple where G is a set of tableaux, R is a binary relation on G, i.e. $R \subseteq G \times G$, and $T_0 \in G$. Then we say that $\mathscr{G}$ is a *tree* obtained from the tableau T_0 if

(i) for any $T, T' \in G$, $\langle T, T'\rangle \in R$ iff $T = T'$ or there is a sequence $T_1, \ldots, T_n$ of elements of G such that $T = T_1$, $T_n = T'$ and $T_1 r_1 T_2 r_2 \ldots T_{n-1} r_{n-1} T_n$ for some rules $r_1, \ldots, r_{n-1}$ belonging to FI;

(ii) $\langle T_0, T\rangle \in G$ for every $T \in G$.

It is easily seen that if $\langle G, R, T_0\rangle$ is a tree, then $\langle G, R\rangle$ is a partially ordered set.

The triple

(6) $\quad \mathscr{G}_0 = \langle G_0, R|G_0, T_0\rangle$

is said to be a *branch* of the tree $\mathscr{G} = \langle G, R, T_0\rangle$ if $G_0 \subseteq G$, $T_0 \in G_0$, $R|G_0 = (G_0 \times G_0) \cap R$, and G_0 is a chain in $\mathscr{G}$ and there is no chain in $\mathscr{G}$ including G_0 as a proper subset.

A tableau $T(\Gamma|Q)$ is *closed* iff $\Gamma \cap Q \neq \varnothing$ or $1 \in Q$ or $0 \in \Gamma$. A branch is *closed* iff there is a closed tableau on this branch. A tree is *closed* iff its every branch is closed; otherwise, it is *open*.

2.2. THEOREM (Fitting [2], p. 30). *A formula X is a thesis of* INT *iff there is a closed tree obtained from $T(\varnothing|X)$. Equivalently X is not a thesis of* INT *iff every tree obtained from $T(\varnothing|X)$ is open.*

2.3. The system $FI(1)$. The formalism of transforming rules has been introduced for the sake of tableau proofs. In order to apply it to the method of rejection we have introduced suitable modifications.

For $Q \neq \emptyset$, the rules rC_2 and rN_2 are called *cancellation* rules.

Let us notice that applying a cancellation rule to the tableau

(7) $T(\Gamma | A_1 \to B_1, \ldots, A_i \to B_i, \sim B_{i+1}, \ldots, \sim B_n, Q)$

we may receive one of the following tableaux:

(8) $T(\Gamma, A_1 | B_1), \ldots, T(\Gamma, A_i | B_i), T(\Gamma, B_{i+1} | \emptyset), \ldots, T(\Gamma, B_n | \emptyset)$.

It is easily seen that we do not loss any generality if we state that applications of cancellation rules to the tableau

(9) $T(\Gamma | C_1, \ldots, C_n, Q)$

in which $C_1, \ldots, C_n$ are implications or negations, yield one of the following tableaux:

(10) $T(\Gamma | C_1), \ldots, T(\Gamma | C_n)$.

This suggests that in the situation under consideration the tree may be further constructed in different m ways if on this tree there appears tableau (7) (or, generally, tableau (9)). We would like to simplify this situation and to propose some new notions concerning conjunctive branchings.

Let us take under account the set $FI(1)$ of rules obtained from the set FI by replacing the rules rN_2 and rC_2 by the following ones:

$rN_2(1)$ $\dfrac{T(\Gamma | \sim A)}{T(\Gamma, A | \emptyset)}$ $rC_2(1)$ $\dfrac{T(\Gamma | A \to B)}{T(\Gamma, A | B)}$

and by adding the following new rule

r_d $\dfrac{T(\Gamma | C_1, \ldots, C_n, Q)}{T(\Gamma | C_1), \ldots, T(\Gamma | C_n)}$

where $C_1, \ldots, C_n$ are implications or negations.

Hence the rule r_d has the following form

(11) $\dfrac{T}{T_1, \ldots, T_n}$.

Of each of the tableaux $T_1, \ldots, T_n$ we say that it is *derived* from the tableau T by the rule r_d, and this fact will be noted by the formula

(12) $T r_d T_i$ for $i = 1, \ldots, n$.

We say that this rule causes *conjunctive* branchings.

Tableau proofs connected with the rules $FI(1)$ have the form of trees, as in the preceding section. The definition of a *tree* is analogous to the one above: in its formulation one must only replace FI by $FI(1)$. So far we had diadic trees. Now the rule r_d may give trees with nodes forking into several branches.

Moreover, we adopt the following two principles of transforming tableaux:

Firstly, we apply all rules but r_d until there is no further possibility of applying them or till we arrive at a terminal tableau (in the sense defined below).

Secondly, we apply the rule r_d provided we have already applied all possible other rules, i.e. if all formulas in the left and the right column of the tableau under consideration have the simplest form.

We define a tableau $T = T(\Gamma|Q)$ to be *terminal* if one of the two following conditions is satisfied:

(a) $\Gamma \cap Q \neq \emptyset$ or $1 \in Q$ or $0 \in \Gamma$

(b) $\Gamma \cap Q = \emptyset$ and there is no tableau $T_1 = T(\Gamma_1|Q_1)$ such that $\langle T, T_1 \rangle \in R$, $\Gamma \subsetneq \Gamma_1$ and $Q \subsetneq Q_1$.

If T fulfils (a), then T is called a *closed terminal* tableau. If T obeys (b), it is called an *open terminal* tableau.

Now, let $\mathscr{G} = \langle G, R, T_0 \rangle$ be a tree obtained from a tableau T_0. The triple

(13) $\mathscr{G}_1 = \langle G_1, R|G_1, T_0 \rangle$

is called a *sub-tree* of $\mathscr{G}$ if $G_1 \subseteq G$, $T_0 \in G_1$ and $\mathscr{G}_1$ is a maximal tree not containing alternative branchings, i.e. there is no set $G_2 \subseteq G$ properly extending G_1 such that $\langle G_2, R|G_2, T_0 \rangle$ is a tree not containing alternative branchings.

One can notice that sub-trees just defined correspond, in some sense, to branches (relative to the set FI), and that sub-trees may contain conjunctive branchings.

If $\langle G_1, R|G_1, T_0 \rangle$ is a sub-tree, then by a *branch* of this sub-tree we understand any triple $\langle G_2, R|G_2, T_0 \rangle$ such that $G_2 \subseteq G$ and G_2 is a chain which contains no chain as a proper subset. Therefore any branch of a sub-tree does not contain any branchings.

Now we define some notions related to the system based on the set of rules $FI(1)$.

A branch of a sub-tree is *closed* if its terminal tableau is closed.

A branch of a sub-tree is *open* if its terminal tableau is open.

A sub-tree is *closed* if some of its branches is closed.

A sub-tree is *open* if all its branches are open.

A tree is *closed* if all its sub-trees are closed. And finally, a tree is *open* if some of its sub-trees is open.

It is easily seen that the definition of a tree obtained from a given tableau T determines a family of trees. By the adopted convention concerning the manner of applying the rules $FI(1)$ it follows that either all trees obtained from T are closed or all these trees are open. In other words, if a tree obtained from T is closed (resp. open), then every tree obtained from T is closed (resp. open). Considering our applications of the tableau method only one thing is important: whether a tree obtained from a given tableau T is closed or open. Hence we may assume that the inscription $\langle G, R, T\rangle$ denotes arbitrary but fixed tree obtained from the tableau T.

By Theorem 2.2 and foregoing explanation we have

2.4. THEOREM. *A formula X is a thesis of* INT *iff the tree $\langle G, R, T(\emptyset|X)\rangle$ is closed.*

3. Ł-decidability of the intuitionistic propositional calculus

In this section we use the formalism defined by the rules $FI(1)$.

Define At to be the set containing all atomic formulas plus the two constants 0 and 1. Let L be the set consisting of all theses of INT plus the constant 1.

3.1. DEFINITION. Recalling that W is the set of all propositional formulas of INT including 1 and 0 we put

$$W1 = \{X \in W: X = A \to B \text{ for some } A, B \in W\}$$

$$W2 = \{X \in W: X = \sim A \text{ for some } A \in W\}.$$

3.2. DEFINITION. The operation * fulfils the following conditions:

(a) $\emptyset^* = \{\{\emptyset\}\}$
(b) if $A \in At \cup W1 \cup W2$, then $\{A\}^* = \{\{A\}\}$
(c) $\{A \vee B\}^* = \{\Delta: \Delta = \Gamma \cup Q \text{ and } \Gamma \in \{A\}^* \text{ and } Q \in \{B\}^*\}$
(d) $\{A \wedge B\}^* = \{A\}^* \cup \{B\}^*$
(e) $\{A_1, \ldots, A_n\}^* = \{A_1 \vee \ldots \vee A_n\}$.

Let L^{-1} be the set of all propositional formulas which are not theses of INT, i.e. let

(14) $L^{-1} = W - L.$

The notation $\vdash A$ means that A is a thesis of INT, i.e. $A \in L$. The notation $\dashv A$ is to be understood that A is a *rejected* formula in the system we are going to define just now.

We adopt the following axiom and rules of rejection.

Ax1 $\quad \dashv p \quad$ where p is a fixed atomic formula

$r_1^{-1} \quad \dfrac{\vdash A \to B, \quad \dashv B}{\dashv A}$

$r_2^{-1} \quad \dfrac{\dashv e(A)}{\dashv A} \quad$ where $e(A)$ is a substitution instance of A.

In order to define the third rule r_3^{-1} we need some auxiliary definitions:

(15) $A = A_1 \wedge \ldots \wedge A_n$ with $A_1, \ldots, A_n \in At \cup W1 \cup W2$ and $n \geqslant 1$,
(16) $|A| = \{A_1, \ldots, A_n\}$,
(17) $|A_r| = \{X \in W\colon \sim X \in |A|$ or $X \to Y \in |A|,\ Y \in W\}$,
(18) $B = B_1 \vee \ldots \vee B_m$ with $B_1, \ldots, B_m \in W1 \cup W2$ and $m \geqslant 1$,
(19) $|B| = \{B_1, \ldots, B_m\}$,
(20) $C = C_1 \vee \ldots \vee C_k$ with $C_1, \ldots, C_k \in At$ and $k \geqslant 0$,
(21) $|C| = \{C_1, \ldots, C_k\}$.

Moreover, we define two conditions to be satisfied by the premisses of the rule r_3^{-1}:

(*) $|A| \cap |C| = \emptyset$
(**) there is $\Gamma \in |A_r|^*$ such that $\Gamma \subseteq |B| \cup |C|$.

Now we are in position to formulate the third rule: Let A, B, C be formulas of the form (15), (18) and (20) respectively. Then

$$r_3^{-1} \qquad \frac{\dashv A \to B_1, \ldots, \dashv A \to B_m}{\dashv A \to B \vee C} \quad \text{provided } (*) \text{ and } (**).$$

We define $Ł$ to be the set of all rejected formulas of INT, i.e. those formulas which can be derived from the axiom Ax1 by means of rules $r_1^{-1}, r_2^{-1}, r_3^{-1}$. It is obvious that $L \cup L^{-1} = W$ and $L \cap L^{-1} = \emptyset$. Hence in order to prove $Ł$-decidability of INT it suffices to show that $Ł = L^{-1}$.

It is obvious that the following lemma holds.

3.3. LEMMA.
(i) *$p \in L^{-1}$, where p is a variable.*
(ii) *If $X \to Y \in L$ and $Y \in L^{-1}$, then $X \in L^{-1}$.*
(iii) *If $e(X)$ is a substitution instance of X and $e(X) \in L^{-1}$, then $X \in L^{-1}$.*

3.4. LEMMA. *Let the formulas A, B and C fulfil conditions* (15)–(21). *Moreover, assume that conditions* (*) *and* (**) *hold. Then for some $i \leqslant m$ $A \to B_i \in L$ if $A \to B \vee C \in L$.*

PROOF. By Theorem 2.4 it follows that the tree obtained from the tableau $T(\emptyset | A \to B \vee C)$ is closed. Hence the tree obtained from the tableau

(22) $T_0 = T(A_1, \ldots, A_n | B_1, \ldots, B_m, C_1, \ldots, C_k)$

must be closed too.

From (*) it follows that the formulas $C_1, \ldots, C_k$ do not cause the tableau T_0 to be closed. And by (**) we conclude that one may apply to T_0 the rule r_d. This means that we have already applied all the possible rules but r_d to all the tableaux from which T_0 has been obtained. Equivalently, applying to T_0 all possible rules but $\mathrm{r_d}$ one gets a sub-tree with the tableau T_0 on one of its nodes. The application of $\mathrm{r_d}$ gives conjunctive branchings, and the tableaux

(23) $T_1 = T(A_1, \ldots, A_n|B_1), \ldots, T_m = T(A_1, \ldots, A_n|B_m)$

are the first ones appearing on every branch.

If every tree obtained from the tableaux $T_1, \ldots, T_m$ were opened, then the tree $\langle G, R, T_0\rangle$ obtained from T_0 would be opened too. Therefore one of the trees obtained from $T_1, \ldots, T_m$ is closed. Thus

(24) $A \to B_i \in L$ for some $1 \leqslant i \leqslant m$.

By Lemmas 3.3 and 3.4 we get

3.5. LEMMA. $Ł \subseteq L^{-1}$.

If Γ is a finite set of formulas, then $\bigwedge \Gamma$ and $\bigvee \Gamma$ stand for a conjunction and alternation of all formulas belonging to Γ. In particular, if $\Gamma = \emptyset$, then

(25) $\bigwedge \emptyset = 1$ and $\bigvee \emptyset = 0$.

We define the formula $\bigwedge \Gamma \to \bigvee Q$ to be a *characteristic formula* of the tableau $T(\Gamma|Q)$.

3.6. LEMMA. *If* $T(\Gamma|Q)\mathrm{r}T(\Gamma_1|Q_1)$ *where* $\mathrm{r} \in FI(1) - \{\mathrm{r_d}\}$, *then* $(\bigwedge \Gamma \to \bigvee Q) \to (\bigwedge \Gamma_1 \to \bigvee Q_1) \in L$.

PROOF. By inspection of the rules $FI(1)$.

Let $\overline{\bigwedge \Gamma} \to \overline{\bigvee Q}$ be the formula obtained from $\bigwedge \Gamma \to \bigvee Q$ by replacing each atomic subformula which belongs to Γ by 1 and each atomic subformula which does not belong to Γ by 0.

3.7. LEMMA. *If* $T(\Gamma|Q)$ *is an open terminal tableau, then* $(\overline{\bigwedge \Gamma} \to \overline{\bigvee Q}) \to 0 \in L$.

PROOF. It is obvious that

(26) $(\overline{\bigwedge \Gamma} \to \overline{\bigvee Q}) \in L$ or $\sim(\overline{\bigwedge \Gamma} \to \overline{\bigvee Q}) \in L$.

Since $T(\Gamma|Q)$ is open we get from (26) that

(27) $\sim(\overline{\bigwedge \Gamma} \to \overline{\bigvee Q}) \in L$

that is

(28) $(\overline{\bigwedge \Gamma} \to \overline{\bigvee Q}) \to 0 \in L$.

3.8. LEMMA. $L^{-1} \subseteq Ł$.

PROOF. Let $X \in L^{-1}$. Then the tree obtained from $T_0 = T(\emptyset|X)$ is open, i.e. there is a sub-tree of this tree, say $\mathscr{G} = \langle G, R, T_0\rangle$, which is open sub-tree. Hence every branch of this sub-tree ends with an open terminal tableau.

We shall prove that

(29) If $T(\Gamma|Q) \in G$, then $\dashv \bigwedge \Gamma \to \bigvee Q$.

By Lemma 3.7 and by the application of the rules r_1^{-1}, r_2^{-1} we get

(30) If $T(\Gamma|Q)$ is an open terminal tableau, then $\dashv \bigwedge \Gamma \to \bigvee Q$.

By Lemma 3.6 and rule r_1^{-1} it follows that

(31) If $T(\Gamma_1|Q_1)$, $T(\Gamma_2|Q_2) \in G$, $\dashv \bigwedge \Gamma_2 \to \bigvee Q_2$ and $T(\Gamma_1|Q_1) r T(\Gamma_2|Q_2)$ for some $r \in FI(1) - \{r_d\}$, then $\dashv \bigwedge \Gamma_1 \to \bigvee Q_1$.

Finally, assume that $T, T_1, \ldots, T_m \in G$ where $T = T(\Gamma_0|Q_0)$, $T_1 = T(\Gamma_1|Q_1), \ldots, T_m = T(\Gamma_m|Q_m)$, and $Tr_d T_1, \ldots, Tr_d T_m$. On the strength of the foregoing assumptions concerning the use of the rule r_d, we conclude that there are numbers m, n and k such that

$$\Gamma_0 = \{A_1, \ldots, A_n\} \quad (\Gamma_0 = |A|)$$
$$Q_0 = \{B_1, \ldots, B_m, C_1, \ldots, C_k\} \quad (Q_0 = |B| \cup |C|)$$
$$\Gamma_1 = \Gamma_2 = \ldots = \Gamma_m = |A|$$
$$Q_1 = \{B_1\}, \ldots, Q_m = \{B_m\},$$

where A_i, B_i and C_i are formulas of the form (15)–(21) satisfying condition (**). Since $\mathcal{G}$ is an open sub-tree, it follows that condition (*) is satisfied too. Hence by r_3^{-1} we get

(32) If $\dashv \bigwedge \Gamma_1 \to \bigvee Q_1, \ldots, \dashv \bigwedge \Gamma_m \to \bigvee Q_m$, then $\dashv \bigwedge \Gamma_0 \to \bigvee Q_0$.

Thus we have shown that

(a) the characteristic formula of an open terminal tableau is a rejected formula;
(b) if T' originates from T by the application of a rule different from the rule r_d and if the characteristic formula of T' is rejected, so is the characteristic formula of T;
(c) if $T_1, \ldots, T_m$ originate from T by the application of r_d and all the characteristic formulas of $T_1, \ldots, T_m$ are rejected, so is the characteristic formula of T.

Summing up we conclude that the characteristic formula of every tableau of the tree $\langle G, R, T_0 \rangle$ is a rejected formula, i.e. we have just proved (29). And this implies that

(33) $\dashv X$,

which ends the proof of 3.8.

By Lemmas 3.4 and 3.8 it follows our main theorem.

3.9. THEOREM. *The intuitionistic propositional calculus is Ł-decidable.*

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DEPARTMENT OF LOGIC AND
METHODOLOGY OF SCIENCES
WROCŁAW UNIVERSITY
ul. SZEWSKA 36
50-139 WROCŁAW

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