Refutation systems in the finite

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Abstract

Refutation systems are deductive systems intended to derive the non-valid, i.e. (semantically) refutable formulae of a given logical system. The goal of this paper is to present some refutation systems on finite semantic structures and establish some basic facts about them. In particular, we develop generic refutation systems for modal logics and for first-order theories that are semantically determined by single finite structures or by classes of finite structures, for arbitrary first-order languages.

1 Introduction

Refutation systems are deductive systems intended to derive the non-valid, i.e. (semantically) refutable formulae of a given logical system. A (syntactic) refutation system R typically consists of refutation axioms and of refutation rules. We say that a formula φ is *refutable in* R, usually denoted by $\mathsf{R} \dashv \varphi$, iff it is derivable in R from the refutation axioms by using the refutation rules. For more on refutation systems see e.g. [11].

In this paper we develop generic refutation systems for modal logics and first-order theories that are semantically determined by single finite structures or classes of finite structures, for arbitrary first-order languages. We prove that these systems are sound and refutation-complete, i.e. they derive all, and only, non-valid formulas in their respective semantics.

Generally, such results could be regarded just as technical exercises in translating semantic into syntactic refutations, but they do have their own value. A good example is FO in the finite $\mathsf{FO}^{\mathsf{fin}}$, which, by Trakhtenbrot's theorem (see e.g. [3]), has no recursive axiomatization. However, the non-validities of $\mathsf{FO}^{\mathsf{fin}}$ are recursively enumerable. Therefore, they can, in principle, be axiomatized by refutation systems, and such systems have both technical and methodological importance. We briefly discuss that issue further in the paper.

Some related works on refutation systems in the finite include:

- In [8] probably the first explicitly developed axiomatic refutation system for FO, for languages without equality and function symbols, was presented and proved sound and complete for falsifiability in finite FO models.

- In [13] a Gentzen type refutation system with a natural set of rules for FO with equality but again without function symbols, is proposed for deriving falsifiable sequents ('antisequents'). That system was proved sound and complete for falsifiability in finite FO models, and an application for loop detecting in Prolog programs is proposed in that paper.

- Some general results about refutation systems for for modal logics corresponding to (classes) of finite models were obtained in [4], [10], [12], [2]. The latter three of these papers deal with modal algebras, whereas the first one is based on Kripke frames.

2 Preliminaries

2.1 Basic logical notation and terminology

We assume that the reader has basic background in modal logic, including Kripke frames and models, Kripke semantics, incl. satisfiability and validity of modal formulae, and bisimulations and n-bisimulations between Kripke models. For these, refer e.g. to [6]. We only provide here basic notation and terminology of modal logic used further.

We denote the classical propositional logic by PL. Let ML be the standard modal propositional logic extending PL, with formulas generated from a set of atomic propositions $AT = \{p_1, p_2, ...\}$ (for which we use p, q as metavariables) by using the constant $\bot$ and the connectives $\neg, \wedge, \vee, \rightarrow, \Box$. We also define, as usual, $\top := \neg\bot$, $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ and $\Diamond\varphi := \neg\Box\neg\varphi$. If Φ, Ψ are finite sets of formulas, then $\Phi \longrightarrow \Psi$ stands for $\bigwedge \Phi \rightarrow \bigvee \Psi$. We put, as usual, $\bigvee\emptyset := \bot$ and $\bigwedge\emptyset := \top$. We also write Φ, Ψ for $\Phi \cup \Psi$. A *(Kripke) frame* is a pair $\mathcal{F} = \langle W, R\rangle$, where W is a nonempty set of points (worlds) and R is a binary relation on W. A *(Kripke) model* is a pair $\langle \mathcal{F}, V\rangle$, where $\mathcal{F}$ is a frame and $V : VAR \mapsto \mathcal{P}(W)$ is a *valuation*, extended to all formulas in the standard way. We denote truth of a modal formula φ at a world w in a Kripke model $\langle \mathcal{F}, V\rangle$ by $\langle \mathcal{F}, V\rangle, w \models \varphi$. The formula φ is *valid in* $\langle \mathcal{F}, V\rangle$, denoted $\langle \mathcal{F}, V\rangle \models \varphi$, if $\langle \mathcal{F}, V\rangle, w \models \varphi$ for each $w \in \mathcal{F}$; φ is *valid in* $\mathcal{F}$, denoted $\mathcal{F} \models \varphi$, if $\langle \mathcal{F}, V\rangle \models \varphi$ for every valuation V. Finally, φ is *valid*, denoted $\models \varphi$, if it is valid in every frame.

We will also be working with first-order languages with any (finite) signature, containing relational, constant and functional symbols. Let us fix any such language L.

- First-order terms and formulas in L are defined as usual. We will denote constants by $c, d, ...$, variables by $u, v, x, y, ...$, and terms by s, t – all possibly with indices.

- An *atomic formula* is either $t_1 = t_2$, where t_1, t_2 are terms, or $Rt_1, ..., t_n$, where R is an n-ary relation symbol and $t_1, ..., t_n$ are terms.

- The set of FO formulas FOR is generated from the atomic formulas by the connectives $\rightarrow, \wedge, \vee, \neg$ and the quantifiers $\forall, \exists$. A *literal* is an atomic formula or a negated one. We write $t_1 \neq t_2$ for $\neg t_1 = t_2$.

Let $\varphi = \varphi(x_1, ..., x_n)$ by any FO formula in the language L with free variables amongst $x_1, ..., x_n$, let $\mathcal{A}$ be an FO structure for L, and let α be an assignment in $\mathcal{A}$ such that $\alpha(x_1) = a_1, ..., \alpha(x_n) = a_n$. Then we write $\mathcal{A} \models \varphi[a_1, ..., a_n]$ for $\mathcal{A} \models \varphi[\alpha]$. If φ is a sentence, we write simply $\mathcal{A} \models \varphi$ (read: "$\mathcal{A}$ satisfies φ") instead of "$\mathcal{A} \models \varphi[\alpha]$ for some (hence. every) α".

For other standard concepts and notation of FO used here see e.g. in [3].

Let us fix any complete standard axiomatic system $\mathsf{Ax}_{\mathsf{FO}}$ for FO (see e.g. [1] or [5]). A formula φ is $\mathsf{Ax}_{\mathsf{FO}}$*-provable* iff φ has a derivation in $\mathsf{Ax}_{\mathsf{FO}}$. By completeness, φ is $\mathsf{Ax}_{\mathsf{FO}}$*-provable* iff φ is true in all structures under all assignments.

2.2 Refutation systems: basic concepts

We now define basic concepts by modifying and extending some terminology and results from [4], [7]. In general, by a sequent we mean $\Phi \bowtie \Psi$, where

$\bowtie \in \{\vdash, \dashv\}$. However, here we will only be dealing with sequents $\bowtie \varphi$, that is, sequents with $\Phi = \emptyset$ and Ψ consisting of a single formula φ.

By a *refutation rule* we mean a rule of the following form.

$$\frac{\vdash \varphi_1, \ldots, \vdash \varphi_m,\ \dashv \psi_1, \ldots, \dashv \psi_n}{\dashv \gamma}$$

Global interpretation (valid/non-valid):

If each φ_i is valid and each ψ_j is non-valid, then γ is non-valid.

A typical example is Łukasiewicz's rule *Reverse modus ponens*:

$$\frac{\vdash \varphi \to \psi, \dashv \psi}{\dashv \varphi}$$

We now introduce a *local refutation rule* as follows:

$$\frac{\Vdash_1 \varphi_1, \ldots, \Vdash_m \varphi_m,\ \dashv\!\|_1\psi_1, \ldots, \dashv\!\|_n\psi_n}{\dashv\!\|_0\gamma}$$

It has a *local interpretation* (true/false):

Let $\langle \mathcal{F}, V \rangle$ be a finite model. If each φ_i is true at a world $u_i \in W$ and each ψ_j is false at a world $w_j \in W$, then γ is false at the world $w_0 \in W$.

A *refutation system* is a set R of refutation rules, some of them possibly local. Refutation rules with no premises are called *refutation axioms* and we write $\dashv \varphi$, resp. $\dashv\!\|_i\varphi$.

We say that a formula φ is R-*refutable* (or just *refutable*) iff there is an R -derivation for $\dashv \varphi$, that is, a sequence $S_1, ..., S_t$, where S_t is $\dashv \varphi$ (resp. $\dashv\!\|_i\varphi$) and every S_i is either a refutation axiom or has the form $\vdash \vartheta$ (resp. $\Vdash_i \vartheta$) or is obtained from some preceding formulas by a refutation rule.

Let Φ be a set of formulas in some standard logical language L. We say that a refutation system R is *(sound and) complete for* Φ if for any formula φ of the language L the following holds: φ *is* R-*refutable iff* $\varphi \notin \Phi$.

The refutation system R is complete for a given modal logic $\mathbf{L}$ if it is complete for the set of valid formulae of $\mathbf{L}$. Likewise, for first-order theories.

3 Refutation systems for modal logics in the finite

3.1 Preliminaries: characteristic modal formulas for pointed finite Kripke models

Consider any pointed finite Kripke model $(\mathcal{M}, w)$. For any $n \in \mathbb{N}$, it can be characterised up to n-bisimulation by a single *characteristic modal formula of modal depth* n, denoted $\chi^n_{[\mathcal{M},w]}$ and defined inductively on n as follows (cf. [6]).

The formula $\chi^0_{[\mathcal{M},w]}$ is the conjunction of all $p \in AT$ that are true in w and all $\neg p$ for those that are false at w. It characterises the propositional type of w in $\mathcal{M}$.

Now, suppose the formulae $\chi^n_{[\mathcal{M},u]}$ are defined, for all $u \in \mathcal{M}$. Then we define

$$\chi^{n+1}_{[\mathcal{M},w]} := \chi^0_{[\mathcal{M},w]} \wedge \underbrace{\bigwedge_{(w,u)\in R} \Diamond \chi^n_{[\mathcal{M},u]}}_{\textit{forth}} \quad \wedge \quad \underbrace{\Box \bigvee_{(w,u)\in R} \chi^n_{[\mathcal{M},u]}}_{\textit{back}}.$$

Thus, $\chi^{n+1}_{[\mathcal{M},w]}$ describes all successors of w in $\mathcal{M}$ up to depth n.

Let ML^n denote the set of modal formulae of modal depth up to n. Two pointed Kripke models $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$ are called *modally n-equivalent*, denoted $(\mathcal{M}, w) \equiv_{\mathsf{ML}^n} (\mathcal{M}', w')$, if they satisfy the same modal formulae of modal dept up to n. The following claim (cf. [6]) captures the use of the characteristic modal formulae for our purposes.

Proposition 3.1. *For every two pointed Kripke models $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$ defined on a finite set of atomic propositions the following are equivalent:*

1. $\mathcal{M}', w' \models \chi^n_{[\mathcal{M},w]}$.
2. $(\mathcal{M}, w) \equiv_{\mathsf{ML}^n} (\mathcal{M}', w')$.

3.2 Generic refutation systems for modal logics of finite frames

Consider a finite Kripke frame $\mathcal{F} = \langle W, R \rangle$, where $W = \{w_1, \dots, w_k\}$.

Denote by $\mathbf{K}_{\mathcal{F}}$ the normal modal logic of all validities in $\mathcal{F}$.

Let $P = \{p_1, \ldots, p_k\}$ be a set of fixed distinct propositional variables and $\mathcal{F}^P = \langle \mathcal{F}, V^P \rangle$ be the Kripke model where $V^P(p_i) = \{w_i\}$, for $i = 1, \ldots, k$, and $V^P(q) = \emptyset$ for any other propositional variable q. Further, let $\Sigma_\vee(P)$ be the set of all substitutions that replace any propositional variable by a disjunction of variables from P.

Let $\mathsf{Ax}_{\mathbf{K}}$ be any fixed complete axiomatic system for the basic normal modal logic $\mathbf{K}$.

Theorem 3.2. *The refutation rule schema*

$$\textbf{\textit{Ref}}_{\mathcal{F}} \qquad \frac{\vdash_{\mathsf{Ax}_{\mathbf{K}}} \chi^n_{[\mathcal{F}^P,w]} \to \neg\sigma(\varphi)}{\dashv \varphi}$$

where $w \in W$, $\varphi \in \mathsf{ML}^n$, and $\sigma \in \Sigma_\vee(P)$, is complete for the modal logic $\mathbf{K}_{\mathcal{F}}$.

Proof. Suppose φ is semantically refutable in $\mathbf{K}_{\mathcal{F}}$, i.e. falsifiable at some world w for some valuation V in $\mathcal{F}$. Then there is a substitution $\sigma \in \Sigma_\vee(P)$ such that $\sigma(\varphi)$ fails at w in $\mathcal{F}^P$, viz. the one defined by $\sigma(q) := \bigvee\{p_i \in P \mid w_i \in V(q)\}$ for each $q \in AT$.

Therefore, $\sigma(\varphi)$ fails at every pointed Kripke model $(\mathcal{M}, w)$ defined on P which is n-equivalent to $(\mathcal{F}^P, w)$. By Proposition 3.1, these are precisely the pointed models $(\mathcal{M}, w)$ defined on P that satisfy $\chi^n_{[\mathcal{F}^P,w]}$. We can assume that $\mathcal{M}$ is extended arbitrarily over AT.

Thus, $M, w \models \neg\sigma(\varphi)$ for any pointed model (M, w) such that $M, w \models \chi^n_{[\mathcal{F}^P,w]}$.

Then $\models \chi^n_{[\mathcal{F}^P,w]} \to \neg\sigma(\varphi)$. By completeness of $\mathsf{Ax}_{\mathbf{K}}$, it follows that $\vdash_{\mathbf{K}} \chi^n_{[\mathcal{F}^P,w]} \to \neg\sigma(\varphi)$.

Therefore, by applying the rule $\mathbf{Ref}_{\mathcal{F}}$ we derive $\dashv \varphi$. $\square$

Theorem 3.2 provides a generic, simple and complete refutation system for every modal logic $\mathbf{K}_{\mathcal{F}}$, which only employs derivations in $\mathsf{Ax}_{\mathbf{K}}$. However, it has some clear shortcomings, which are the price to pay for the generality:

- Involves infinitely many (one for each n) and rather complex refutation rules. Moreover, they have no subformula property.
- Does not build the refutable formulae step-by-step, but derives them at once.

- Most essential: to derive $\dashv \varphi$ it requires guessing the falsifying substitution σ, which amounts to finding a refuting valuation for φ in $\mathcal{F}$.

3.3 An alternative construction of a refutation system for the modal logic of a finite frame

Again, consider a finite Kripke frame $\mathcal{F} = \langle W, R \rangle$, where $W = \{w_1, \ldots, w_k\}$. We will use the notation from Section 3.2. and will now build in a uniform way a different refutation system $\mathbf{Ref}(\mathcal{F})$ for $\mathcal{F}$, by adding 'local' rules for every state of $\mathcal{F}$, in order to generate all formulae falsifiable at that state from the formulae falsifiable at its successors in the frame.

The refutation system $\mathbf{Ref}(\mathcal{F})$ employs local refutation rules, using an additional auxiliary symbol $\dashv\!\Vdash$ which will be used in combination with any substitution $\sigma \in \Sigma_{\vee}(P)$. Intuitively, $w \dashv\!\Vdash_{\sigma} \varphi$ means that φ is falsified at the state w by a valuation syntactically described by the substitution σ.

The system $\mathbf{Ref}(\mathcal{F})$ and consists of several schemes of refutation rules listed below, where:

- $\vdash_{\mathsf{PL}}$ is any complete deduction system for PL.
- $\vdash_{\Box\mathsf{PL}}$ refers to derivability in $\vdash_{\mathsf{PL}}$, but applied to modal formulae of ML, where all subformulas of the type $\Box\psi$ are treated as atomic symbols (recall that $\Diamond$ is assumed defined as $\neg\Box\neg$).
- σ is any substitution in $\Sigma_{\vee}(P)$.

The refutation rules of $\mathbf{Ref}(\mathcal{F})$:

1. For every $w_i, w \in W$ and a classical propositional formula β,

$$\mathbf{Ref}^0(w_i) \quad \frac{\vdash_{\mathsf{PL}} p_i \to \sigma(\neg\beta)}{w_i \dashv\!\Vdash_{\sigma} \beta}$$

   and

$$\mathbf{Ref}^1(w) \qquad \frac{w \dashv\!\Vdash_{\sigma} \psi, \ \vdash_{\Box\mathsf{PL}} \varphi \to \psi}{w \dashv\!\Vdash_{\sigma} \varphi}$$

2. For every $w, w' \in W$ such that wRw':

$$\mathbf{Ref}^{\Box}(w, w') \qquad \frac{w \dashv\!\Vdash_{\sigma} \varphi, \ w' \dashv\!\Vdash_{\sigma} \psi}{w \dashv\!\Vdash_{\sigma} \varphi \vee \Box\psi}$$

3. For every $w \in W$ and $\{u_1, ..., u_m\}$ as its set of all R-successors of w:

$$\mathbf{Ref}^{\Diamond}(w) \qquad \frac{w \dashv\!\!|_{\sigma}\, \varphi, \quad u_1 \dashv\!\!|_{\sigma}\, \psi, ..., u_m \dashv\!\!|_{\sigma}\, \psi}{w \dashv\!\!|_{\sigma}\, \varphi \vee \Diamond\psi}$$

4. Finally, for every $w \in W$:

$$\mathbf{Ref}(w) \quad \frac{w \dashv\!\!|_{\sigma}\, \varphi}{\dashv \varphi}$$

Theorem 3.3. *The refutation system* $\mathbf{Ref}(\mathcal{F})$ *is complete for the logic* $\mathbf{K}_{\mathcal{F}}$.

Proof. First, we note, by routine induction on derivations in $\mathbf{Ref}(\mathcal{F})$, that whenever $w \dashv\!\!|_{\sigma}\, \varphi$ is derivable then φ is falsifiable in $\mathcal{F}$ at the state w by the valuation V_σ defined by $V_\sigma(q) = \{w_i \in W \mid p_i$ occurs as a disjunct in $\sigma(q)\}$. Therefore, if $\dashv \varphi$ then φ is falsifiable in $\mathcal{F}$ at w by the valuation V_σ, hence φ is semantically refutable in $\mathbf{K}_{\mathcal{F}}$.

For the converse, first we define for any valuation V in $\mathcal{F}$ the substitution $\sigma_V \in \Sigma_\vee(P)$ as in the proof of Thm 3.2, viz. $\sigma_V(q) := \bigvee\{p_i \in P \mid w_i \in V(q)\}$ for each $q \in AT$. Note that for every modal formula φ and $w \in W$:
$\langle \mathcal{F}, V\rangle, w \models \varphi$ iff $\langle \mathcal{F}, V^P\rangle, w \models \sigma_V(\varphi)$.

Now, we will prove by induction on $d \in \mathbb{N}$ that for every $\varphi \in \mathsf{ML}^d$, if φ is falsifiable in $\mathcal{F}$ at some state w by a valuation V then $w \dashv\!\!|_{\sigma_V} \varphi$ is derivable in $\mathbf{Ref}(\mathcal{F})$, hence $\dashv \varphi$ is derivable in $\mathbf{Ref}(\mathcal{F})$, by $\mathbf{Ref}(w)$.

First, let $d = 0$ and $\varphi \in \mathsf{ML}^0$ be a propositional formula falsifiable by V at some $w_i \in W$. Then $\langle \mathcal{F}, V^P\rangle, w_i \models \sigma_V(\neg\varphi)$, hence $\models p_i \to \sigma_V(\neg\varphi)$, so $\vdash_{\mathsf{PL}} p_i \to \sigma_V(\neg\varphi)$. Therefore $w_i \dashv\!\!|_{\sigma_V} \varphi$ by $\mathbf{Ref}^0(w)$.

Suppose the claim holds for all formulae in ML^d for some $d \in \mathbb{N}$.

Now, let $\varphi \in \mathsf{ML}^{d+1}$ be falsifiable at some state $w \in \mathcal{F}$ by some valuation V, i.e., $\langle \mathcal{F}, V\rangle, w \models \neg\varphi$. Note that φ is propositionally equivalent to a conjunction of formulae of the type

$$\theta = \beta \vee \Diamond\varphi_1 \vee \ldots \vee \Diamond\varphi_m \vee \Box\psi_1 \vee \ldots \vee \Box\psi_n,$$

where $m, n \geq 0$ and $\varphi_1, \ldots, \varphi_m, \psi_1, ..., \psi_n \in \mathsf{ML}^d$.

Then, at least one of these conjuncts θ is false at w in $(\mathcal{F}, V)$. We fix it.

We will show that $w \dashv\!\!|_{\sigma_V} \theta$ is derivable in $\mathbf{Ref}(\mathcal{F})$.

To begin with, we have $\langle \mathcal{F}, V^P\rangle, w \models \sigma_V(\neg\beta)$, hence $w \dashv\!\!|_{\sigma_V} \beta$ is derivable in $\mathbf{Ref}(\mathcal{F})$, by using $\mathbf{Ref}^0(w)$ as in the case $d = 0$.

Next, each ψ_i is falsified in $\langle \mathcal{F}, V\rangle$ at some successor w^i of w and each φ_j is falsified in $\langle \mathcal{F}, V\rangle$ at all successors u of w. By the inductive hypothesis, it follows that $w^i \dashv\!|_{\sigma_V} \psi_i$, for each $i = 1, ..., n$, and $u \dashv\!|_{\sigma_V} \varphi_j$ for each $j = 1, ..., m$ and each successor u of w, are all derivable in $\mathbf{Ref}(\mathcal{F})$.

Now, by applying $\mathbf{Ref}^{\Diamond}(w)$ consecutively, for $\psi = \varphi_1, \ldots, \varphi_m$, we derive $w \dashv\!|_{\sigma_V} \beta \lor \Diamond\varphi_1 \lor \ldots \lor \Diamond\varphi_m$.

Then, likewise, by applying $\mathbf{Ref}^{\Box}(w, w')$ consecutively, for $w' = w^1, ..., w^m$, we eventually derive $w \dashv\!|_{\sigma_V} \theta$ in $\mathbf{Ref}(\mathcal{F})$.

Finally, note that $\vdash_{\Box\mathsf{PL}} \varphi \to \theta$, hence $w \dashv\!|_{\sigma_V} \varphi$ is derivable in $\mathbf{Ref}(\mathcal{F})$, by $\mathbf{Ref}^{1}(w)$, which completes the induction, and the entire proof. $\Box$

3.4 On refutation systems for modal logics with finite model property

Each of the refutation systems described in sections 3.2 and 3.3 extends to the modal logic $\mathbf{K}_{\mathcal{T}}$ of any r.e. set $\mathcal{T}$ of finite Kripke frames, by adding the respective refutation rules for each frame $\mathcal{F} \in \mathcal{T}$.

In particular, using this generic construction, a complete generic refutation system can likewise be produced for any (finitely axiomatized) normal modal logic with finite frame property[1]. Interestingly, even finite axiomatisation is not needed (and, possibly, not even existing) when the class of all finite frames of the logic is explicitly known. An important case in point is Medvedev's logic (see [9]), the decidability of which is not known yet, but, being defined on a class of simply and explicitly defined finite frames, it has a recursively enumerable set of non-validities. Actually, the following simple refutation system is complete for this set (see [9]).

Refutation axiom: $\bot$ (the false).

Refutation rules:

- (**RS**) (If a substitution instance of φ is refutable, so is φ.)
- ($\mathbf{RMP}_{KP}$) $\vdash_{KP} \varphi \to \psi, \dashv \psi / \dashv \varphi$
- (Here $\vdash_{KP}$ means provability in the Kreisel-Putnam logic.)
- (**RD**) $\dashv \varphi, \dashv \psi / \dashv \varphi \lor \psi$

[1] The finite axiomatization is only needed in order to be able to recognise the finite frames of that logic.

4 Refutation system for FO in the finite

4.1 Refutation system for the FO theory of a single finite model

Fix any FO language L with finite signature.

Definition 4.1. *Let $n \in \mathbb{N}^+$. For any list of distinct variables $\overline{x} = \{x_1, ..., x_n\}$ we define:*

1. $\Gamma_n(\overline{x}) = \{x_i = x_j : 1 \leq i < j \leq n\}$,
2. $\delta_n(\overline{x}) = \forall x_{n+1}(x_{n+1} = x_1 \vee ... \vee x_{n+1} = x_n)$.
3. $\Theta_n(\overline{x})$ *is the set of* ***basic atomic formulae of*** L ***over*** $\overline{x}$*, consisting of all formulae of the form:*

   (a) $x_i = x_j$*, or*

   (b) $c = x_j$ *for any constant symbol* $c \in L$*, or*

   (c) $Ry_1, ..., y_k$ *for any* k*-ary relational symbol* $R \in L$*, or*

   (d) $f(y_1, ..., y_k) = y$ *for any* k*-ary functional symbol* $f \in L$*,*

   with variables $y_1, ..., y_k, y \in \{x_1, ..., x_n\}$.

Definition 4.2. *Consider a finite L-structure $\mathcal{A}$ with universe $A = \{a_1, ..., a_n\}$. We define the FO sentence $\chi_\mathcal{A}$ that describes $\mathcal{A}$ up to isomorphism (see e.g. [3, p. 13], where it is denoted $\varphi_\mathcal{A}$), as follows:*

$$\chi_\mathcal{A} := \exists x_1 ... \exists x_n \Big(\delta_n(\overline{x}) \wedge \bigwedge\{\psi \mid \psi \in \mathsf{POS}(\mathcal{A})\} \wedge \bigwedge\{\neg\psi \mid \psi \in \mathsf{NEG}(\mathcal{A})\}\Big),$$

where $\overline{x} = \{x_1, ..., x_n\}$ is a list of distinct variables,
$\overline{x} := \overline{a}$ is the assignment α in $\mathcal{A}$ such that $\alpha(x_1) = a_1, ..., \alpha(x_n) = a_n$,
$\mathsf{POS}(\mathcal{A}) = \{\psi \mid \psi \in \Theta_n(\overline{x})$ *and* $\mathcal{A} \models \psi[\overline{x} := \overline{a}]\}$*, and*
$\mathsf{NEG}(\mathcal{A}) = \{\psi \mid \psi \in \Theta_n(\overline{x})$ *and* $\mathcal{A} \models \neg\psi[\overline{x} := \overline{a}]\}$
Note that every formula $x_i = x_j$, where $i \neq j$, is in $\mathsf{NEG}(\mathcal{A})$.

Hereafter we fix any complete deductive system $\mathsf{Ax_{FO}}$ for FO.

Lemma 4.3. *Let $\mathcal{A}$ be a finite structure and φ be a sentence which is not true in $\mathcal{A}$. Then*

$$\vdash_{\mathsf{Ax_{FO}}} \varphi \rightarrow \neg\chi_\mathcal{A}$$

Proof. The sentence $\neg\varphi$ is true in every FO structure isomorphic to $\mathcal{A}$, i.e. satisfying $\chi_{\mathcal{A}}$. Therefore, $\models \chi_{\mathcal{A}} \to \neg\varphi$, hence $\models \varphi \to \neg\chi_{\mathcal{A}}$. Then the claim follows by completeness of $\mathsf{Ax}_{\mathsf{FO}}$. □

Theorem 4.4. *Let $\mathcal{A}$ be any finite FO structure. Then the refutation system consisting of*

$$\textbf{\textit{Ref}}_{\mathcal{A}} \qquad \frac{\vdash_{\mathsf{Ax}_{\mathsf{FO}}} \chi_{\mathcal{A}} \to \neg\varphi}{\dashv \varphi}$$

and ***Reverse generalisation****:*

$$\textbf{\textit{RG}} \quad \frac{\dashv \forall x\varphi}{\dashv \varphi}$$

is complete for the FO theory TH($\mathcal{A}$) of $\mathcal{A}$.

Proof. First, let φ be a sentence which is not true in $\mathcal{A}$. Then $\vdash_{\mathsf{Ax}_{\mathsf{FO}}} \chi_{\mathcal{A}} \to \neg\varphi$, by Lemma 4.3. Then, by applying $\mathbf{Ref}_{\mathcal{A}}$, we obtain $\dashv \varphi$.

Now, let $\varphi = \varphi(\overline{x})$ be any formula with a tuple of free variables $\overline{x} = \langle x_1, ..., x_k\rangle$, which is not true in $\mathcal{A}$ and let $\forall\overline{x}\varphi$ be a universal closure of φ. Then, $\mathcal{A} \not\models \forall\ \overline{x}\varphi$. Hence, by the argument above, $\dashv \forall\overline{x}\varphi$. By using the rule **RG** repeatedly, we eventually obtain $\dashv \varphi$. □

Just like with modal logic, the refutation system above extends readily to the FO theory of any r.e. set $\mathcal{T}$ of finite FO structures, by adding refutation rules $\mathbf{Ref}_{\mathcal{A}}$ for each $\mathcal{A} \in \mathcal{T}$. In particular, a complete refutation system can thus be produced for all FO formulae that are not valid in the finite. That system, however, involves infinitely many refutation rules $\mathbf{Ref}_{\mathcal{A}}$ and is not practically very useful. In the next subsection we will develop an alternative, purely syntactic refutation system for the FO on all finite structures.

4.2 Refutation system $\mathsf{Ref}^{fin}_{\mathsf{FO}}$ for FO in the finite

Let $\mathsf{FO}^{\mathsf{fin}}$ be the set of all first-order formulas true in all finite structures under all assignments, and let $\mathsf{RFO}^{\mathsf{fin}}$ be its complement, i.e. the set of first-order formulas falsifiable in some finite structure under some assignment.

By Trakhtenbrot's theorem, $\mathsf{FO}^{\mathsf{fin}}$ is not recursively enumerable, hence it cannot have a complete recursive axiom system. However, $\mathsf{RFO}^{\mathsf{fin}}$ is recursively enumerable and therefore it can be recursively axiomatized. In this section we give a refutation system for the set of the formulas of FO logic that are not valid in the finite.

First, some preliminaries. Hereafter, $\vdash$ stands for provability in some fixed complete deductive system $\mathsf{Ax}_{\mathsf{FO}}$ for FO.

We now introduce the following refutation system $\mathsf{Ref}_{\mathsf{FO}}^{fin}$ for FO.

Refutation axioms. (Recall $\Gamma_n(\overline{x})$ and $\Theta_n(\overline{x})$ from Definition 4.1.)

$$\dashv \Phi, \delta_n(\overline{x}) \longrightarrow \Gamma_n(\overline{x}) \cup \Psi$$

where $n \geq 1$ and $\Phi \cup \Psi$ is a finite set of basic atomic formulae from $\Theta_n(\overline{x})$, such that:

1. $\Phi \cap \Psi = \emptyset$.
2. $\Phi \cup \Psi$ does not contain formulae of the type $x_i = x_j$.
3. For every constant symbol $c \in L$, exactly one formula of the type $c = x_i$ is in Φ.
4. For every k-ary functional symbol $f \in L$ and $y_1, ..., y_k \in \{x_1, ..., x_n\}$, exactly one formula of the type $f(y_1, ..., y_k) = x_i$ is in Φ.

Refutation rules:

Reverse modus ponens (**RMP**):

$$\frac{\vdash \varphi \to \psi \qquad \dashv \psi}{\dashv \varphi}$$

Reverse generalisation (**RG**):

$$\frac{\dashv \forall x \varphi(x)}{\dashv \varphi(x)}$$

Lemma 4.5. *If φ is a refutation axiom, then $\varphi \in \mathsf{RFO}^{\mathsf{fin}}$.*

Proof. Let $\varphi = \Phi, \delta_n(\overline{x}) \longrightarrow \Gamma_n(\overline{x}) \cup \Psi$ be a refutation axiom over $x_1, ..., x_n$. Consider the n-element structure $\mathcal{A}$ with domain $A = \{a_1, ..., a_n\}$, where:

1. For every constant symbol $c \in L$, its interpretation $c^{\mathcal{A}}$ is the (unique) $a_i \in A$ such that $c = x_i$ is in Φ.

2. For every k-ary functional symbol $f \in L$ its interpretation $f^{\mathcal{A}}$ is defined as follows: for every tuple $\langle a_{i_1}, ..., a_{i_k} \rangle \in A^k$, $f^{\mathcal{A}}(a_{i_1}, ..., a_{i_k})$ is the (unique) $a_i \in A$ such that $f(x_{i_1}, ..., x_{i_k}) = x_i$ is in Φ.

3. For every k-ary relational symbol $R \in L$ its interpretation $R^{\mathcal{A}}$ is defined as follows: for every tuple $\langle a_{i_1}, ..., a_{i_k} \rangle \in A^k$, $R^{\mathcal{A}}(a_{i_1}, ..., a_{i_k})$ holds iff $Rx_{i_1}, ..., x_{i_k} \in \Phi$.

Consider the assignment α in $\mathcal{A}$, where $\alpha(x_i) := a_i$, for $i = 1, ..., n$. Then:

- $\mathcal{A} \models \delta_n(\overline{x})[\alpha]$.
- If $\theta \in \Phi$ then $\mathcal{A} \models \theta[\alpha]$.
- If $\theta \in \Gamma_n(\overline{x}) \cup \Psi$ then $\mathcal{A} \not\models \theta[\alpha]$.

Hence, $\mathcal{A} \not\models \varphi[\alpha]$. □

Lemma 4.6. *Let φ be any formula. Then $\forall x_i \varphi \in \mathsf{RFO}^{\mathsf{fin}}$ iff $\varphi \in \mathsf{RFO}^{\mathsf{fin}}$.*

Proof. Straightforward. □

Theorem 4.7. *Let φ be any formula. Then $\varphi \in \mathsf{RFO}^{\mathsf{fin}}$ iff $\dashv \varphi$.*

Proof. ($\Leftarrow$) holds since the refutation axioms are not in $\mathsf{FO}^{\mathsf{fin}}$, and the set $\mathsf{RFO}^{\mathsf{fin}}$ is closed under the refutation rules, by Lemma 4.6.

($\Rightarrow$) Let $\varphi \in \mathsf{RFO}^{\mathsf{fin}}$. First, suppose φ is a sentence. Then φ is falsified by some finite structure $\mathcal{A}$, with universe $\{a_1, ..., a_n\}$.

By Lemma 4.3 , we have $\vdash \varphi \to \neg \chi_{\mathcal{A}}$. (1)

Note that $\vdash \neg \chi_{\mathcal{A}} \leftrightarrow \forall x_1 ... \forall x_n \mathsf{NDIAG}(\mathcal{A})$, where (recall Def. 4.2)

$$\mathsf{NDIAG}(\mathcal{A}) = \mathsf{POS}(\mathcal{A}), \delta_n(\overline{x}) \longrightarrow \mathsf{NEG}(\mathcal{A})$$

Hence, $\vdash \neg \chi_{\mathcal{A}} \to \forall x_1 ... \forall x_n \mathsf{NDIAG}(\mathcal{A})$. (2)

Note that $\mathsf{NDIAG}(\mathcal{A})$ is a refutation axiom. Thus, $\dashv \mathsf{NDIAG}(\mathcal{A})$. Hence, $\dashv \forall x_1 ... \forall x_n \mathsf{NDIAG}(\mathcal{A})$, by repeated application of **RMP**, using $\vdash \forall x \varphi \to \varphi$.

Then, $\dashv \neg \chi_{\mathcal{A}}$, by (2) and **RMP**.

Therefore, $\dashv \varphi$, by (1) and **RMP**.

Now, let $\varphi \in \mathsf{RFO}^{\mathsf{fin}}$, where φ is any formula, and let $\forall \overline{x} \varphi$ be a universal closure of φ. Then, $\forall \overline{x} \varphi \in \mathsf{RFO}^{\mathsf{fin}}$. Hence, by the argument above, $\dashv \forall \overline{x} \varphi$. By using the rule **RG** repeatedly, we eventually obtain $\dashv \varphi$. □

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