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A COMPLETE SET OF AXIOMS FOR LOGICAL FORMULAS INVALID IN SOME FINITE DOMAIN[1])

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§ 1. Introduction

By the GÖDEL completeness theorem we know that the set Γ, consisting of those formulas of the first-order predicate calculus valid in every non-empty domain of individuals, is the same as the set of formulas derivable from a suitably chosen collection of postulates (axioms and rules of inference). From the nature of the postulates it is readily seen that the set is recursively enumerable[2]); moreover, by CHURCH's result, this set is not recursive — implying the unsolvability of the decision problem for provability and validity in the first-order predicate calculus (KLEENE [1], pp. 313, 432, 393). Also, by well known results ([1], p. 284 Theorem VI (c), p. 306 Theorem XIV), the complement $\overline{\Gamma}$ is not recursively enumerable. Since, by usual standards as to what is meant by an axiomatization, it is necessary that the set of derivable formulas be recursively enumerable, it then follows that $\overline{\Gamma}$, the set of invalid formulas, is not axiomatizable. When obtainable, an axiomatization is desirable as constituting a perspicuous, syntactical characterization.

A certain amount of interest attaches to the set Φ consisting of those formulas valid in all finite non-empty domains of individuals. In 1950 TRAHTENBROT [2] (see also CRAIG [3]) showed that Φ is not recursive, so that the decision problem for validity (provability not being defined for Φ) is unsolvable. That provability cannot be defined for Φ comes from the result that Φ is not recursively enumerable, as may be seen from TRAHTENBROT's result as follows:

There is an effective procedure for testing a first-order formula for validity in a domain of a fixed finite number of individuals ([1], § 36). Let us refer to the procedure which tests formulas for validity in domains of k individuals as $\mathfrak{P}(k)$. Further let $F_1, F_2, \ldots, F_j, \ldots$ be an enumeration of all formulas of the predicate

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[2]) Given or presupposing, an effective correlation between formulas and natural numbers (GÖDEL numbering), a set of formulas is recursively enumerable if the set of its correlated GÖDEL numbers is recursively enumerable. A similar definition applies for recursiveness. In such an effective correlation one can for any formula of the set effectively determine its correlated number and, conversely given any number it can be effectively determined if it is the GÖDEL number of a formula and of which formula it is the GÖDEL number.

calculus and let $\langle 1, 1\rangle, \langle 1, 2\rangle, \ldots, \langle k, j\rangle, \ldots$ be an enumeration of all ordered pairs of natural numbers. Then by recording the negative results obtained by applying $\mathfrak{P}(k)$ to F_j (i.e. the results in which the decision is that the formula is invalid) in the enumerated order of all pairs $\langle k, j\rangle$, there is successively generated, hence recursively enumerated, all formulas invalid in some finite domain. Thus $\bar{\Phi}$, the complement of Φ, is recursively enumerable. Since a set is recursive if it and its complement are recursively enumerable ([1], pp. 284, 306) and since by TRAHTENBROT's result Φ is not recursive, while the complement $\bar{\Phi}$ is recursively enumerable, it then follows that Φ is not recursively enumerable, and hence not axiomatizable.

On the other hand, $\bar{\Phi}$ is not recursive (since Φ is not recursive) and hence, like the set Γ, it has the interesting property of being recursively enumerable but not recursive. It would seem worthwhile to have an axiomatization for $\bar{\Phi}$— a simple set of syntactically characterized initial formulas and rules of inference from which the entire set $\bar{\Phi}$ is derivable; the class of axioms and the relation of immediate consequence are, of course, to be recursive. Such a formal system together with a proof of completeness of the axioms we shall present in this paper.

If one is not too particular as to the kind of axiomatization then, as shown by HERMES [4], any recursively enumerable set of formulas can be axiomatized— in fact by using any member of the set of formulas as a starting formula and with a single two-place recursive predicate as the relation of immediate consequence (rule of inference). This relation of HERMES' is defined as follows: given the recursive function φ which enumerates the set of formulas (let us identify these with their GÖDEL numbers, so that φ is a number-theoretic function) then b is defined to be an immediate consequence of a if

$$\exists x(x \leqq \max(a, \varepsilon y(\varphi(y) > \cdot a)) \wedge b = \varphi(x)),$$

that is to say if b is either $\varphi(0)$, or $\varphi(1)$, or $\ldots$, $\varphi(k)$, where k is a if each of these φ values is $\leqq a$, and if not then k is the least integer for which $\varphi(k) > a$. Thus the number of formulas which are immediate consequences of a formula can, as a function of the formula, grow arbitrarily large. Such a rule of inference seems quite foreign to normal notions of proof; moreover the connection between the syntactic structure of the premises and the conclusion seems highly indirect. In contrast, the type of formalization we present in § 3 will maintain a high degree of intuitive content and resemble, as best as can be done in the circumstances, our usual notions.

Concerning the nature of this formalization and its consistency and completeness a number of comments may be made.

(I) The BERNAYS rule of substitution (of a formula for a predicate with arguments) does not preserve finite invalidity—thus the simple atomic formula Fa is finitely invalid, but its BERNAYS substitution instance $Ga \vee \neg Ga$ is not—and hence this rule is unavailable to us.

(II) The rule of detachment, i.e. modus ponens, is not useful in the calculus of finite invalidity since from the finite invalidity of $P \supset Q$, or for that matter $P \vee Q$, that of Q follows without the need for an additional premise. Although the rule: from $P \vee Q$ infer Q, is "valid" in that it leads from finitely invalid to finitely invalid formulas, we shall not use it in our formalization. The system we shall set up will have the character, first delineated by GENTZEN, that derivable formulas are built up during the course of a derivation and logical symbols once introduced are not subsequently lost. The particular form of the system used (rather than other versions first entertained) was suggested to the author by a reading of K. SCHÜTTE's [10] and results in a completeness proof of unparalleled simplicity.

(III) There are interesting sidelights in connection with consistency and completeness. For the formal system to be presented it is the case that for some formulas P, both P and $\neg P$ are derivable; however, it is not the case that all formulas are derivable. In particular only finitely invalid formulas are derivable (semantic consistency). Parallel results concerning completeness are that the addition as an axiom of an underivable formula need not result in the derivability of all formulas; and for any formula P, either P or $\neg P$ (or both!) are derivable. More generally, any finitely invalid formula is derivable (semantic completeness).

(IV) The three rules of inference of the formal system to be presented are all one-premise rules, so that a derivation consists of a linear chain of formulas starting with an axiom and having each formula as an immediate consequence of its predecessor. The axioms are non-tautologous quantifier-free formulas, Rule 1 can only be used once and must precede any uses of Rules 2 or 3, and all uses of Rule 2 can be made to precede any uses of Rule 3. Thus the system admits of a normal form for derivations.

(V) One might wonder why it is not possible to use, when suitably modified, one of the reductio ad absurdum proof procedures (e.g. HINTIKKA [7], BETH [8] or QUINE [9], appendix) which are used to show universal validity, also as a direct proof of finite invalidity. These procedures are such that, when applied to a formula $\neg A$, they lead to a contradiction when A is universally valid and otherwise a model satisfying $\neg A$. It might be thought that the formulas invalid in some finite domain would be selected by those cases in which there was a finite satisfying model for $\neg A$. However if a formula is invalid in a domain of k individuals it is also invalid in every domain of a larger number of individuals (see Theorem 2.3 below). Thus the proof procedures mentioned may furnish infinite rather than finite models in the cases of interest, and hence not distinguish between (a) formulas invalid in a finite domain and (b) formulas valid in all finite domains but invalid in an infinite domain. And, indeed, this is the case — it is easy to furnish examples of formulas invalid in finite domains for which the above mentioned procedures do not come to a halt after a finite number of steps (cf. the example in QUINE [9], pp. 256–257).

§ 2. Notational and semantic preliminaries

Our *formulas* are those of the usual first-order predicate calculus built up from *prime formulas* (predicates with argument places filled by individual variables[1]) in customary fashion by means of

Connectives: $\neg$ not), $\vee$ (or), $\wedge$ (and),

Quantifiers: $\exists$ (existential), $\forall$ (universal).

As with HILBERT-BERNAYS [11] we shall use typographically distinct letters for free and bound variables ("variable" shall always mean "individual variable") The free variables are $a_1, a_2, \ldots, a_n, \ldots$ and the bound variables are $x, y, z, \ldots$ with and without subscripts. Bound variables, e.g. x, may have *free occurrences* in an expression (as distinct from a formula) A, namely those occurrences of x not within the scope of a quantifier $\exists x$ or $\forall x$. Vacuous quantifiers are excluded. By $A(a/y)$ we shall designate the result of replacing all free occurrences of y in A by a, not precluding that A may have occurrences of a to begin with.

Finite disjunctions and conjunctions will often be written in "closed" form, e.g.

$$\bigvee_{i=1}^{n} A[a_i] \quad \text{for} \quad A[a_1] \vee A[a_2] \vee \ldots \vee A[a_n]$$

$$\bigwedge_{i=1}^{n} A[a_i] \quad \text{for} \quad A[a_1] \wedge \ldots \wedge A[a_n];$$

where for repeated alternations and conjunctions we assume the convention of association to the left.

The *positive* and *negative parts* and the *positive* and *negative quantifier occurrences* of an expression are defined recursively as follows[2]):

(i) H is a positive part of H,

(ii) if $\neg A$ is a positive (negative) part of H, then A is a negative (positive) part of H,

(iii) if $A \vee B$, or $A \wedge B$, is a positive (negative) part of H, then A and B are positive (negative) parts of H,

(iv) if $\exists x A$ is a positive (negative) part of H, then A is a positive (negative) part of H and $\exists x$ is a positive (negative) quantifier occurrence,

(v) if $\forall x A$ is a positive (negative) part of H then A is a positive (negative) part of H and $\forall x$ is a n e g a t i v e (p o s i t i v e) quantifier occurrence.

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[1]) It will be convenient for us to assume that all predicates have one or more argument places so that an atomic formula has at least one free variable.

[2]) Here we follow HERBRAND [12] except in interchanging the roles of positive and negative quantifiers.

In this definition we are considering only specific occurrences of expressions, so that an expression could have occurrences some of which are positive and some negative in a given formula. It may be noted, though we make no use of this fact here, that when a formula is converted to prenex normal form, all the positive quantifier occurrences in the original formula come out to be existential quantifiers in the prefix, and all the negative ones come out to be universal quantifiers.

Following SCHÜTTE [10], we employ the notation $H_+[A]$ for an arbitrary formula having a particular occurrence of an expression A as a positive part; and, in the same context, $H_+[B]$ shall be the result of replacing this occurrence of A by B (assuming the result to be a formula). Similarly for $H_-[A]$ and $H_-[B]$—where the minus sign indicates that the expression in brackets is a negative part of the formula.

The *elementary parts* of a formula are those quantifications (i.e. expressions of the form $\exists xA$ or $\forall xA$) and prime formulas which are not within the scope of a quantifier. Supposing that quantifications which can be transformed into each other by alphabetic changes of bound variables are identified, we say that a formula is *tautologously valid* (for short, *tautologous*) if, on replacing each distinct elementary part (uniformly in all of its occurrences) by distinct propositional variables, the resulting formula is a tautology. It is well-known (cf. HILBERT-ACKERMANN [5], p. 128–129) that a quantifier-free formula is valid, and this in a domain of k individuals, where k is the number of distinct free variables of the formula, if and only if the formula is tautologous. Hence

Theorem 2.1. *A quantifier-free formula having k distinct free variables is invalid, and this in a domain of k individuals, if and only if it is non-tautologous* (*i.e. not a tautology*).

The phrase "invalid in a domain of k individuals" will be shortened to "k-invalid".

To each formula H we associate a quantifier-free formula $H^{(k)}$ called its *k-transform* defined as follows (cf. KLEENE [1], § 37): replace successively each quantification of the form $\exists xA$ by $\bigvee_{i=1}^{k} A(a_i/x)$ and each $\forall xB$ by $\bigwedge_{i=1}^{k} B(a_i/x)$. The resulting k-transform (which is independent of the order in which the quantifications are replaced) has then all the free variables $a_1, \ldots, a_k$ in addition to those originally present, and none other. By virtue of Theorem 2.1 and the definition of validity in a domain of k individuals we have

Theorem 2.2. *A formula having at most the free variables $a_1, \ldots, a_k$ is k-invalid if and only if its k-transform is non-tautologous.*

If a formula is valid in a domain of k individuals it is also valid in every non-empty smaller domain (cf. HILBERT-ACKERMANN [5], p. 122); hence

Theorem 2.3. *If a formula is k-invalid then it is also k'-invalid for any $k' \geqq k$.*

The content of the next theorem is that, relative to the partial ordering of implication, formulas are "non-decreasing" functions of positive parts and "non-increasing" functions of negative parts (cf. HERBRAND [12], p. 35).

Theorem 2.4. *Let P be a quantifier-free formula in which there is singled out a particular occurrence of a formula E. If E' implies E, then $P_+[E']$ implies $P_+[E]$ if E is a positive part of P, and $P_-[E]$ implies $P_-[E']$ if E is a negative part of P.*

Proof. By induction on the number of logical connectives in P other than those in E. We use "$=$" as short for "is the same as" and "$\rightarrow$" for "implies".

Basis:

Case 1. P is E. Here $P = P_+[E] = E$, and $E' \rightarrow E$ is the same as $P_+[E'] \rightarrow P_+[E]$.

Case 2. P is $\neg E$. Here $P = P_-[E] = \neg E$ and $P_-[E'] = \neg E'$. Thus if $E' \rightarrow E$, then $\neg E \rightarrow \neg E'$, that is $P_-[E] \rightarrow P_-[E']$.

Induction Step:

Case 1. P is of the form $Q \vee R$.

Sub-case 1a. E is a positive part of Q. (The argument is similar if E is a positive part of R). Here $P = P_+[E] = Q_+[E] \vee R$. By hypothesis of induction $Q_+[E'] \rightarrow Q_+[E]$. Whence $(Q_+[E'] \vee R) \rightarrow (Q_+[E] \vee R)$, which is our $P_+[E'] \rightarrow P_+[E]$.

Sub-case 1b. E is a negative part of Q. Here $P = P_-[E] = Q_-[E] \vee R$. By hypothesis of induction $Q_-[E] \rightarrow Q_-[E']$, from which our result readily follows.

Case 2. P is of the form $Q \wedge R$. The argument here is similar to that of Case 1.

Case 3. P is of the form $\neg Q$.

Sub-case 3a. E is a positive part of P. Then $P = P_+[E] = \neg Q_-[E]$. By hypothesis of induction $Q_-[E] \rightarrow Q_-[E']$ so that $\neg Q_-[E'] \rightarrow \neg Q_-[E]$.

Sub-case 3b. E is a negative part of P. The argument here is similar to that of sub-case 3a.

This completes our proof.

A *k-instance* of an expression $A = A(x, y, \ldots, z)$ is a formula obtained from A as follows: replace the free occurrences of x in A by a_i for some $i = 1, \ldots, k$; similarly for $y, \ldots, z$; then form the k-transform of this formula.

Theorem 2.5. *Consider a particular occurrence of an expression A in a formula H. In forming the k-transform of H this particular occurrence will yield a number (k^n, if n is the number of distinct bound variables with free occurrences in A) of k-instances. If the occurrence of A is a positive part of H, then each of its k-instances will be positive parts of $H^{(k)}$; and if the occurrence is a negative part of H, then its k-instances in $H^{(k)}$ will be negative parts of $H^{(k)}$.*

Proof. By induction on the number of logical operators (connectives and quantifiers) in a formula H other than those in A.

Basis: H is A. Since $n = 0$, here a k-instance of A is just its k-transform and there is nothing to prove, as $H^{(k)}$ is $A^{(k)}$.

Induction Step: H is one of the forms $\neg P$, $P \vee Q$, $P \wedge Q$, $\exists xP$, or $\forall xP$.

Case 1. H is $\neg P$. If A is a positive part of H, then it is a negative part of P and hence, by hypothesis of induction, the k-instances of A are all negative parts of $P^{(k)}$. Whence these k-instances are positive parts of $\neg (P^{(k)}) = H^{(k)}$. A similar argument applies if A is a negative part of H.

Cases 2 and 3. H is $P \vee Q$ or $P \wedge Q$. Similar to Case 1.

Case 4. H is $\forall xP$. To indicate possible dependence of A on x we write P as $P(x, A(x))$, and then

$$H^{(k)} = \bigwedge_{i=1}^{k} (P(a_i, A(a_i/x))^{(k)}.$$

Now assume A is a positive part of $\forall xP$. Then for each i, $A(a_i/x)$ is a positive part of $P(a_i, A(a_i(x))$. By hypothesis of induction, then, each k-instance of $A(a_i/x)$ is a positive part of $(P(a_i, A(a_i/x))^{(k)}$, hence a positive part of $H^{(k)}$. But each k-instance of A is a k-instance of $A(a_i/x)$ for some i, and our conclusion then follows. A similar argument applies if A is a negative part of $\forall xP$.

Case 5. H is $\exists xP$. Similar to Case 4.

§ 3. The formalization of Φ

We now present our axiomatization of the set of finitely invalid formulas. A descriptive explanation follows after the statement of the postulates. For "A is derivable" we use "$\vdash A$"

Axioms. If A is quantifier-free and non-tautologous, then $\vdash A$.

Inference Rules.

Rule 1. Let (for $k > 0$, $m + n > 0$)

$$(1) \qquad M_+(\exists x_1, \ldots, \exists x_n, \forall y_1, \ldots, \forall y_m;\ a_1, \ldots, a_k)$$

be a formula containing only the $n + m$ quantifiers depicted, all of which are to be positive quantifiers, and having no free variables not in the list $a_1, \ldots, a_k$; let

$$(2) \qquad M_+\left(\bigvee_{i_1=1}^{k}, \ldots, \bigvee_{i_n=1}^{k}, \bigwedge_{j_1=1}^{k}, \ldots, \bigwedge_{j_m=1}^{k};\ a_1, \ldots, a_k\right)$$

be the formula which results from (1) by replacing x_r throughout by a_{i_r} and y_s by a_{j_s} ($r = 1, \ldots, n$; $s = 1, \ldots, m$) and replacing the quantifiers $\exists x_r$ by $\bigvee_{i_r=1}^{k}$, $\forall y_s$ by $\bigwedge_{j_s=1}^{k}$. If $\vdash$ (2), then $\vdash$ (1).

Rule 2. (a) If $\vdash H_+[A(a_n/x)]$, then $\vdash H_+[\forall xA]$.

(b) If $\vdash H_-[A(a_n/x)]$, then $\vdash H_-[\exists xA]$. $(n > 0)$

Rule 3. (a) If $\vdash H_+[\forall xA \wedge A(a_n/x)]$, then $\vdash H_+[\forall xA]$.

(b) If $\vdash H_-[\exists xA \vee A(a_n/x)]$, then $\vdash H_-[\exists xA]$. $\quad (n > 0)$

The above rules are to hold for any choice of bound variables in place of those explicitly used provided that the expressions following the $\vdash$-sign are formulas.

Little need be said about the axioms—a quantifier-free non-tautologous formula is invalid in a domain of k individuals for any k greater than or equal to the number of distinct free variables in the formula (by Theorems 2.1 and 2.3).

Inference Rule 1 enables one to introduce positive quantifiers into a quantifier-free derivable formula of specified structure. The rule can thus be used only once, and this before any uses of Rules 2 or 3. Note also that (2) is the k-transform of (1); the intuitive correctness of the rule is thus clear since a formula of the form (1) is invalid in a domain of k individuals if (2) is non-tautologous.

As mentioned earlier, the notation $A(a_n/x)$ permits A to have prior occurrences of a_n. Thus inference Rule 2 is incorrect for the calculus of universal validity; e.g. from $Fa \equiv Fa$ we would not wish to infer $\forall x(Fa \equiv Fx)$.

To illustrate the Rules we consider the example

$$(1) \qquad (Ga_1 \wedge Fa_1a_1) \vee (Ga_1 \wedge Fa_1a_2) \vee (Ga_1 \wedge Fa_2a_1) \vee (Ga_1 \wedge Fa_2a_2),$$

that is $\quad \bigvee_{i=1}^{2} \bigvee_{j=1}^{2} (Ga_1 \wedge Fa_ia_j)$,

which is non-tautologous and hence an axiom; by Rule 1 we may infer

$$(2) \qquad \exists x \exists y (Ga_1 \wedge Fxy).$$

Since Ga_1 is a positive part of (2) we may use Rule 2 to infer

$$(3) \qquad \exists x \exists y (\forall z Gz \wedge Fxy)$$

or, since $Ga_1 \wedge Fxy$ is a positive part of (2),

$$(4) \qquad \exists x \exists y \forall z (Gz \wedge Fxy)$$

or even $\forall z \exists x \exists y (Gz \wedge Fxy)$.

On the other hand from

$$(5) \qquad \bigvee_{i=1}^{2} \bigvee_{j=1}^{2} \neg (Ga_1 \wedge Fa_ia_j)$$

we can infer by Rule 1

$$(6) \qquad \exists x \exists y \neg (Ga_1 \wedge Fxy)$$

but this time Rule 2 does not permit us to infer

$$\exists x \exists y \neg (\forall z Gz \wedge Fxy),$$

since Ga_1 is a negative part of (6); the quantifier $\forall z$ may, however, be placed anywheres to the left of the negation symbol.

As mentioned in the introduction, all the rules of inference are one-premise rules, so that a derivation consists of a sequence of formulas each, except for the first, comes from the preceding formula in the sequence by one of the Rules 1–3. Moreover, all uses of Rule 2 can be made to precede any use of Rule 3:

Theorem 3.1. *Any derivation of a formula of Φ can be converted into another derivation of the same formula, the new derivation having the property that no use of Rule 2 comes after a use of Rule 3.*

Proof. If the derivation is not already in the appropriate form then, since Rule 1 can be applied at most once and any uses of Rules 2 and 3 afterwards, there is a step, s say, which is an application of Rule 3, e.g.

$$\text{step } s: \qquad \frac{H_+[\forall xA \wedge A(a_n/x)]}{H_+[\forall xA]}$$

and step $s+1$ is an application of Rule 2 to $H_+[\forall xA]$. If it should happen that the change at the $(s+1)$-st step does not "affect" the particular occurrence of $\forall xA$ depicted (that is, if the introduction of a quantifier by Rule 2 does not take place in the particular occurrence), then by making the introduction into $H_+[\forall xA \wedge A(a_n/x)]$ by Rule 2 and then use Rule 3 on this result we obtain the same formula as formerly at the $(s+1)$-st step. If the use of Rule 2 does affect the particular $\forall xA$, then by two uses of Rule 2 on $H_+[\forall xA \wedge A(a_n/x)]$ the quantifier may be introduced both in $\forall xA$ and $A(a_n/x)$; then a use of Rule 3 results in the same formula as formerly at step $s+1$.

From this we have

Theorem 3.2. (Normal form for derivations). *To each derivation of a formula of $\bar{\Phi}$ there may be effectively given another derivation of the same formula with the following properties: the first formula is a quantifier-free non-tautologous formula; if the end-formula has any positive quantifiers, these are all introduced at the second step by Rule* 1; *next comes a succession of* 0 *or more uses of Rule* 2, *and finally a succession of* 0 *or more uses of Rule* 3.

§ 4. Consistency of the formalization Φ

For formal logistic systems consistency may be taken either syntactically or semantically. In the case of syntactic consistency essentially two principal types are considered: (i) consistency with respect to negation and (ii) absolute consistency. A system is *consistent with respect to negation* if for no formula P is it the case that P and $\neg P$ are both derivable; it is *absolutely consistent* if not every formula is derivable. For *semantic consistency* one requires that only formulas with a certain semantic property are derivable—in this paper we shall take this property to be finite invalidity.

Since Fa_1 and $\neg Fa_1$, F a one-place predicate, are both derivable (being axioms) it is clear that our formalization does not enjoy consistency with respect to negation. However it does have absolute consistency. This follows immediately from the semantic consistency (to be established in this section): if the system is semantically consistent only finitely invalid formulas are derivable; but a tautologous formula is not finitely invalid and hence not derivable.

The following series of theorems leads to a proof of semantic consistency.

Theorem 4.1. *Each axiom of Φ is finitely invalid.*

Proof. Theorem 2.1.

Theorem 4.2. *If P is finitely invalid and Q results from P by use of Rule* 1, *then Q is finitely invalid.*

Proof. Since P is a quantifier-free formula containing only $a_1, \ldots, a_k$ as free variables and is finitely invalid, it follows by Theorem 2.1 that it is k-invalid. But Q has P as its k-transform; hence by Theorem 2.2 it is also k-invalid, and thus finitely invalid.

Theorem 4.3. *If P is finitely invalid and Q results from P by use of Rule* 2, *then Q is finitely invalid.*

Proof. As before, we use $R^{(k)}$ for the k-transform of R and A_i as short for $A(a_i/x)$. Considering Rule 2a first, if $P = H_+[A_n]$ is finitely invalid, then for some k (which by Theorem 2.3 may be taken $\geqq n$) its k-transform is non-tautologous. The k-transform of P will have one or more k-instances of A_n coming from the particular occurrence of A_n. Call an arbitrary one such k-instance $A_n'^{(k)}$ The k-transform of $Q = H_+[\forall xA]$ will be like that of P except that where $P^{(k)}$ has $A_n'^{(k)}$, $Q^{(k)}$ has $A_1'^{(k)} \wedge A_2'^{(k)} \wedge \ldots \wedge A_k'^{(k)}$. If there is an assignment of truthvalues to the elementary parts of $P^{(k)}$ which renders it false, then since $A_1'^{(k)} \wedge \ldots \wedge A_k'^{(k)}$ implies $A_n'^{(k)}$ it follows, by Theorem 2.4, that this same assignment renders $Q^{(k)}$ false. Thus Q is finitely invalid. The case of Rule 2b is handled dually.

Theorem 4.4. *If P is finitely invalid and Q comes from P by Rule* 3, *then Q is finitely invalid.*

Proof. Similar to Theorem 4.3—note that the k-transform of $\forall xA' \wedge A_n'$ is equivalent to that of $\forall xA'$ when $k \geqq n$.

As an immediate consequence of Theorems 4.1–4.4 we have

Theorem 4.5. (**Semantic consistency.**) *If $\vdash H$, then H is finitely invalid.*

§ 5. Completeness of the axioms for $\overline{\Phi}$

As with consistency, the notion of completeness for a formal system may be taken either in a syntactical or semantical sense. *Completeness with respect to negation* holds if, for any formula P, either P or $\neg P$ is derivable. *Absolute completeness* requires that for any formula P, either P is derivable or the addition of P as an axiom renders the system absolutely inconsistent.

That our formal system is not absolutely complete may be seen from the following argument. Consider the closed formula $\forall x(Fx \vee \neg Fx)$, where F is a one-place predicate, which is certainly valid in every finite domain (in fact in every non-empty domain). If this formula, which is not derivable by our consistency result (Theorem 4.5), were added as an axiom, then we still could not derive every formula—for example a tautology with only free variables. To see this we note that none of the rules of inference can remove all quantifiers if any are present; hence starting from $\forall x(Fx \vee \neg Fx)$ one can never arrive at a quantifier-free formula. The only other possible starting formulas for a proof are the quantifier-free non-tautologies and from these no quantifier-free formulas can be derived—hence no quantifier-free tautology

The formal system is, however, complete with respect to negation. This comes from the semantic completeness (to be established in this section) as follows: given any formula P, it is either finitely invalid, and hence derivable by semantic completeness, or else it is valid in every finite domain. But then $\neg P$ is finitely invalid and so derivable.

We turn now to establishing the semantic completeness of our system. For this purpose we introduce the following operations where, as before, A_j is short for $A(a_j/x)$.

($\mathfrak{O}$1) Beginning at the left-hand end of the formula find the first expression occurrence of the form

$$\forall xA \wedge A_i \wedge A_{i-1} \wedge \ldots \wedge A_1$$

or

$$\exists xA \vee A_i \vee A_{i-1} \vee \ldots \vee A_1 \qquad (i = 0, 1, \ldots, k-1)$$

in which the displayed quantifier is a negative quantifier in the given formula, and $i = 0$ means that $\forall xA$ occurs and is not conjoined with $A_i \wedge \ldots \wedge A_1$ for $i > 0$ (similarly for the existential quantifier). The operation then is to replace such an occurrence by, respectively,

$$\forall xA \wedge A_{i+1} \wedge A_i \wedge \ldots \wedge A_1$$

or

$$\exists xA \vee A_{i+1} \vee A_i \vee \ldots \vee A_1 \qquad (i = 0, 1, \ldots, k-1).$$

($\mathfrak{O}$2) With the same conditions as in ($\mathfrak{O}$1), replace

$$\forall xA \wedge A_k \wedge A_{k+1} \wedge \ldots \wedge A_1 \quad \text{by} \quad A_k \wedge A_{k-1} \wedge \ldots \wedge A_1$$

and

$$\exists xA \vee A_k \vee A_{k-1} \vee \ldots \vee A_1 \quad \text{by} \quad A_k \vee A_{k-1} \vee \ldots \vee A_1.$$

($\mathfrak{O}$3) Use Rule 1 in the reverse direction — that is, to replace an entire formula which is of the form of the conclusion by the formula in the premise.

Theorem 5.1. *To each formula H having no free variable with index (subscript) larger than j, and for each natural number $k \geqq j$, there can be effectively associated by means of ($\mathfrak{O}$1)–($\mathfrak{O}$3) a quantifier-free formula $H^{(k)}$ having exactly $a_1, \ldots, a_k$ as free variables and such that if $H^{(k)}$ is taken as an axiom, then H is effectively derivable from $H^{(k)}$*

Proof. Starting with H, repeatedly apply operations ($\mathfrak{O}1$) or ($\mathfrak{O}2$), whichever applies, until no negative quantifiers remain; the use of ($\mathfrak{O}1$) may increase the number of quantifier occurrences (if any are present in A), but the "depth" of the quantifier occurrences (quantifiers within the scope of quantifiers) is not increased by ($\mathfrak{O}1$) and is reduced by ($\mathfrak{O}2$). When ($\mathfrak{O}1$) and ($\mathfrak{O}2$) have no application then apply ($\mathfrak{O}3$); the resulting formula $H^{(k)}$ is quantifier-free and has exactly the free variables $a_1, \ldots, a_k$. That H is derivable from $H^{(k)}$ follows from the fact that each step in the process of obtaining $H^{(k)}$, when reversed, is justified by a rule of inference-operation ($\mathfrak{O}3$) by Rule 1, ($\mathfrak{O}2$) by Rule 2, and ($\mathfrak{O}1$) by Rule 3.

Theorem 5.2. (Continuation of Theorem 5.1). *If H is k-invalid, then so is $H^{(k)}$.*

Proof. We show that each of the operations ($\mathfrak{O}1$)–($\mathfrak{O}3$) preserves k-invalidity. Consider first ($\mathfrak{O}1$) which changes, for example,

$$H_+[\forall xA \wedge A_i \wedge \ldots \wedge A_1], \tag{1}$$

into

$$H_+[\forall xA \wedge A_{i+1} \wedge \ldots \wedge A_1]. \tag{2}$$

If the k-transform of (1) is non-tautologous (definition of k-invalid) then so is that of (2), for wherever the k-transform of (1) has k-instances of $\forall xA \wedge A_i \wedge \ldots \wedge A_1$, the k-transform of (2) will have corresponding k-instances of $\forall xA \wedge A_{i+1} \wedge \ldots \wedge A_1$, all of which, in both transforms, occur as positive parts (by virtue of Theorem 2.5). Since a k-instance of $\forall xA \wedge A_{i+1} \wedge A_i \wedge \ldots \wedge A_1$ implies the corresponding k-instance of $\forall xA \wedge A_i \wedge \ldots \wedge A_1$ we have, by Theorem 2.5, that the k-transform of (2) implies that of (1). Hence if the k-transform of (1) is non-tautologous so is that of (2). A similar argument applies if the replacement involves a negative existential quantifier. In the case of a replacement by ($\mathfrak{O}2$) the corresponding k-instances of $\forall xA \wedge A_k \wedge \ldots \wedge A_1$ and $A_k \wedge \ldots \wedge A_1$ are logically equivalent and hence the operation preserves k-invalidity. Finally, since the formula which is the premise in Rule 1 is the k-transform of the formula of the conclusion, it follows that ($\mathfrak{O}3$) preserves k-invalidity.

Theorem 5.3. (Semantic Completeness.) *If a formula H is finitely invalid, then $\vdash H$.*

Proof. Suppose H is finitely invalid. Then, for some k_1, H is k'-invalid for all $k' \geqq k_1$ (Theorem 2.3). Let k_2 be the largest subscript of any free variable of H, or 0 if H is closed. Then for $k = \max(k_1, k_2)$, H is k-invalid. By Theorem 5.2, then, so is the associated formula $H^{(k)}$. But $H^{(k)}$ is quantifier-free and has only the free variables $a_1, \ldots, a_k$; hence by Theorem 2.1 it is non-tautologous and therefore an axiom. By Theorem 5.1, H is derivable from $H^{(k)}$; hence derivable. Q.E.D.

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