MATHEMATICS

# REFUTABILITY AND ELEMENTARY NUMBER THEORY

BY

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SUMMARY

It would thus appear that there is not much difference in the case of elementary number theory whether one uses for the underlying logic intuitionism, constructible falsity or refutability. When the domain of discourse is the species of natural numbers, mathematical induction is a very powerful principle which overcomes any deficencies which may be present. Thus if one wishes to obtain a separation of the concepts of intuitionism, constructible falsity and refutability then either one has to give up mathematical induction or else consider some species other than the natural numbers. The latter would probably be more interesting.

## 0. INTRODUCTION

If one has been brought up using truth tables, then a statement of the form "not-$A$" causes no problem; that is "not-$A$" is true iff $A$ is false. But if one wants to go beyond truth tables and yearns for a constructive interpretation then one finds that negative statements can be interpreted in more than one way. Since it is generally accepted that in a constructive interpretation a mathematical proposition $A$ calls for some kind of construction and it is customary to say that the construction is a proof of $A$, we may interpret "not-$A$" in one of the following ways:

(1) there is no construction as required by $A$,

(2) the assumption that there is a construction that proves $A$ leads to a contradiction; or more specifically, there is a construction $\pi$ such that if $p$ is a proof of $A$ then $\pi(p)$ is a proof of an absurdity,

(3) in addition of having the concept of "a construction $c$ proves a formula $B$" there is at hand the concept of "a construction $d$ refutes a formula $C$" and there is a construction which refutes $A$.

The interpretation (1), which is related to Curry's non-demonstrability (see CURRY 1957 page 91) is clearly unsatisfactory from a constructive viewpoint because it asserts a lack of constructions. (2) is the familiar intuitionistic interpretation for negation and there is no doubt that it has been a fruitful interpretation. Nevertheless intuitionistic negation has certain unpleasant characteristics. For example, if one accepts that there is no construction that proves an absurdity (as do most people) then a salient property of the construction $\pi$ that proves "not-$A$" is that when $\pi$ is applied to a particular non-existent construction (namely a proof of $A$) it yields another non-existent construction!

The third interpretation does not have that problem. Of course it has other problems, probably the most damnable being that the number of primitive notions is increased.

The purpose of this paper is to discuss the merits (if any) of taking refutation as the interpretation for negation. The paper is organized as follows: In section 1 we state some of the general properties that should be satisfied by the notion "the construction $c$ is a refutation of the formula $A$". Based on the observations in section 1 we then set up a Gentzen style formalism in section 2. In § 3 we present an adaptation of Kripke models and the (gist of the) completeness of the interpretation via Kripke models is given in § 4. We end with sections 5, 6 where we consider elementary number theory with negation interpreted as refutation.

## § 1. Refutation as a primitive notion

The concept "*the construction c proves the formula A*" is common to most intuitionistic writings. In Heyting 1966 it is considered in informal terms; more formal developments can be found in Kreisel 1962, Troelstra 1968 and in Goodman's thesis 1968. In this section we wish to informally discuss the notion "*the construction c refutes the formula A*". Time will tell whether more formal developments are needed.

First of all we should observe that how a construction $c$ refutes a primitive (atomic) statement depends on the particular discipline that is being considered. Thus the *logically* valid formulae should not be dependent on how the primitive formulae are refuted; what is important is how compound statements are refuted.

Assuming certain basic operations on constructions, such as forming the construction which consists of the ordered pair of two constructions, the following conditions on "*the construction c refutes the formula A*" would probably be acceptable to most people:

i.) *the construction c refutes* $A \,\&\, B$ *iff c is of the form* $\langle i, d\rangle$ *with i either* 0 *or* 1 *and if* $i=0$, *then d refutes A and if* $i=1$ *then d refutes B*,

ii.) *the construction c refutes* $A \vee B$ *iff c is of the form* $\langle d, e\rangle$ *and d refutes A and e refutes B*,

iii.) *the construction c refutes* $A \supset B$ *iff c is of the form* $\langle d, e\rangle$ *and d proves A and e refutes B*,

iv.) *the construction c refutes* $\forall x A(x)$ *iff c is of the form* $\langle a, d\rangle$ *and d refutes* $A(a)$,

vi.) *the construction c refutes* $\exists x A(x)$ *iff c is a general method of construction such that given any individual (i.e. construction) from the species under consideration,* $c(a)$ *(i.e. c applied to a) refutes* $A(a)$.

In order to give the conditions under which a construction $c$ refutes $\neg A$ we first need the following assumptions about constructions and the interpretation of $\neg A$.

I. A construction $c$ proves $\neg A$ iff $c$ refutes $A$.

II. It is decidable whether or not a given construction proves a formula. Similarly it is decidable whether or not a given construction refutes a formula. Of course it is possible for a given construction to neither prove nor refute a formula. On the other hand we do not think that it is reasonable to allow that a given construction $c$ both proves and refutes a formula $A$ (although it is possible that a given construction $c$ proves a formula $A$ and refutes another formula $B$, $A \neq B$).

III. Suppose that $c$ is a construction that refutes $\neg A$. That is $c$ refutes that $A$ is refutable. Hence from $c$ we must be able to extract the information that no construction will ever refute $A$. Since we are assuming that no construction can both refute and prove the same formula and we know from $c$ that no construction will ever refute $A$, it appears reasonable to stipulate that the construction $c$ then proves $A$. (It could also be argued that the only way in which the construction $c$ could encode the information that there will never be found a refutation of $A$ is to encode a proof of $A$).

From the above remarks I, II, and III we then obtain viii. The construction $c$ refutes $\neg A$ iff $c$ proves $A$ [1]).

## § 2. An axiomatization for the calculus

Gentzen style axiomatizations are the formalizations which are simpler to justify on the intended interpretations of the logical operators. For the intuitionistic calculus it is possible to restrict oneself to sequents of the form $\Gamma \rightarrow \Theta$ where $\Gamma$ is a finite (possibly empty) sequence of formulae and $\Theta$ is either empty or consist of exactly one formula. The intended meaning of (the provable) $A_0, \ldots, A_{k-1} \rightarrow \Theta$ is that there is a construction $\pi$ such that if $c_0, \ldots, c_{k-1}$ are constructions that prove $A_0, \ldots, A_{k-1}$ then $\pi(c_0, \ldots, c_{k-1})$ is a construction that proves the formula in $\Theta$. Since it is rather difficult to visualize how we can have a construction that proves the formula occurring in the empty set of formulae we prefer to restrict ourselves to sequents of the form:

$$A_0, \ldots, A_{k-1} \rightarrow B.$$

By **RFC** (refutability calculus) we understand the calculus whose axioms and rules of inference are the following.

*Axiom schema*:

$$B \rightarrow B$$

Structural rules of inference.

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1) It also follows that a construction $c$ proves $\neg\neg A$ iff $c$ proves $A$.

*Thinning*

$$\frac{\Gamma \to B}{A, \Gamma \to B}$$

*Contraction*

$$\frac{A, A, \Gamma \to B}{A, \Gamma \to B}$$

*Interchange*

$$\frac{\Delta, D, C, \Gamma \to B}{\Delta, C, D, \Gamma \to B}$$

*Cut*

$$\frac{\Delta \to C \qquad C, \Gamma \to B}{\Delta, \Gamma \to B}.$$

Logical rules of inference for the quantifiers

($\to \forall$)

$$\frac{\Gamma \to A(\mathbf{b})}{\Gamma \to \forall x A(x)}$$

subject to the usual restriction on the variables.

($\forall \to$)

$$\frac{A(t), \Gamma \to B}{\forall x A(x), \Gamma \to B}$$

($\to \exists$)

$$\frac{\Gamma \to A(t)}{\Gamma \to \exists x A(x)}$$

($\exists \to$)

$$\frac{A(\mathbf{b}), \Gamma \to B}{\exists x A(x), \Gamma \to B}$$

subject to the usual restriction on the variables.

Logical rules of inference for the positive propositional connectives.

($\to \supset$)

$$\frac{A, \Gamma \to B}{\Gamma \to A \supset B}$$

($\supset \to$)

$$\frac{\Delta \to A \qquad B, \Gamma \to C}{A \supset B, \Delta, \Gamma \to C}$$

($\to$ &)

$$\frac{\Gamma \to A \qquad \Gamma \to B}{\Gamma \to A \& B}$$

(& $\to$)

$$\frac{A, C, \Gamma \to B}{A \& C, \Gamma \to B}$$

$(\rightarrow \vee)$

$$\frac{\Gamma \rightarrow A}{\Gamma \rightarrow A \vee B} \qquad \frac{\Gamma \rightarrow B}{\Gamma \rightarrow A \vee B}$$

$(\vee \rightarrow)$

$$\frac{A, \Gamma \rightarrow B \qquad C, \Gamma \rightarrow B}{A \vee C, \Gamma \rightarrow B}.$$

Logical rules of inference for $\neg$

$(\rightarrow \neg\ \neg)$

$$\frac{\Gamma \rightarrow A}{\Gamma \rightarrow \neg\ \neg A}$$

$(\neg\ \neg \rightarrow)$

$$\frac{A, \Gamma \rightarrow B}{\neg\ \neg A, \Gamma \rightarrow B}$$

$(\rightarrow \neg\ \&)$

$$\frac{\Gamma \rightarrow \neg A}{\Gamma \rightarrow \neg (A \& B)} \qquad \frac{\Gamma \rightarrow \neg B}{\Gamma \rightarrow \neg (A \& B)}$$

$(\neg\ \& \rightarrow)$

$$\frac{\neg A, \Gamma \rightarrow C \qquad \neg B, \Gamma \rightarrow C}{\neg (A \& B), \Gamma \rightarrow C}$$

$(\rightarrow \neg \vee)$

$$\frac{\Gamma \rightarrow \neg A \qquad \Gamma \rightarrow \neg B}{\Gamma \rightarrow \neg (A \vee B)}$$

$(\neg \vee \rightarrow)$

$$\frac{\neg A, \neg B, \Gamma \rightarrow C}{\neg (A \vee B), \Gamma \rightarrow C}$$

$(\rightarrow \neg \supset)$

$$\frac{\Gamma \rightarrow A \qquad \Gamma \rightarrow \neg B}{\Gamma \rightarrow \neg (A \supset B)}$$

$(\neg \supset \rightarrow)$

$$\frac{A, \neg B, \Gamma \rightarrow C}{\neg (A \supset B), \Gamma \rightarrow C}$$

$(\rightarrow \neg \forall)$

$$\frac{\Gamma \rightarrow \neg A(t)}{\Gamma \rightarrow \neg \forall x A(x)}$$

$(\neg \forall \rightarrow)$

$$\frac{\neg A(\mathbf{b}), \Gamma \rightarrow B}{\neg \forall x A(x), \Gamma \rightarrow B}$$

subject to the usual restriction on the variables.

$(\rightarrow \neg \exists)$

$$\frac{\Gamma \rightarrow \neg A(\mathbf{b})}{\Gamma \rightarrow \neg \exists x A(x)}$$

subject to the usual restriction on the variables.

$(\neg\, \exists \rightarrow)$

$$\frac{\neg\, A(t),\ \Gamma \rightarrow B}{\neg\, \exists x A(x),\ \Gamma \rightarrow B}.$$

The justification for the axiom schema, structural rules of inference, rules of inference for the quantifiers and the positive propositional connectives can be found in any standard text on intuitionism (for example TROELSTRA 1968). The justification of the rules involving $\neg$ is an immediate consequence of the observations made concerning refutations. For example consider the rule of inference:

$$\frac{A, A_0, \ldots, A_{k-1} \rightarrow B}{\neg\, \neg\, A, A_0, \ldots, A_{k-1} \rightarrow B}.$$

It asserts that if we have a construction $\pi$ such that $\pi(c, c_0, \ldots, c_{k-1})$ is a construction that proves $B$ whenever $c, c_0, \ldots, c_{k-1}$ are proofs of $A, A_0, \ldots, A_{k-1}$ respectively, then there is a construction $\pi^*$ such that $\pi^*(d, d_0, \ldots, d_{k-1})$ proves $B$ whenever $d, d_0, \ldots, d_{k-1}$ proves $\neg\, \neg\, A, A_0, \ldots, \ldots, A_{k-1}$ respectively. But since we have agreed that a construction $d$ proves $\neg\, \neg\, A$ iff $d$ proves $A$ it is clear that for $\pi^*$ we may take $\pi$ itself.

The remaining rules of inference for $\neg$ can be justified in analogous fashion.

A formula $A$ is provable in **RFC** (or **RFC**-provable) iff the sequent $\rightarrow A$ is provable in **RFC**. Using some of the results about the intuitionistic predicate calculus and straightforward applications of the axioms and rules of **RFC** we obtain:

THEOREM 1. (1) *If a formula $A$ is an instance of one of the axiom schemas for the intuitionistic predicate calculus which do not explicitly involve the symbol* $\neg$ (i.e. *axioms* 1a, 1b, 3, 4a, 4b, 5a, 5b, 6 *of* KLEENE 1952 *page* 82) *then $A$ is* **RFC**-*provable.*

(2) *If the formula $A$ contains no occurrence of $\neg$ then $A$ is provable in* **RFC** *iff $A$ is provable in the intuitionistic predicate calculus.*

(3) *The following schema are* **RFC**-*provable*:

$$A \equiv \neg\, \neg\, A$$

$$\neg\, (A \,\&\, B) \equiv \neg\, A \vee \neg\, B$$

$$\neg\, (A \vee B) \equiv \neg\, A \,\&\, \neg\, B$$

$$\neg\, (A \supset B) \equiv A \,\&\, \neg\, B$$

$$\neg\, \forall x A(x) \equiv \exists x \neg\, A(x)$$

$$\neg\, \exists x A(x) \equiv \forall x \neg\, A(x).$$

It follows from theorem 1 that if a formula is **RFC**-provable then it is provable in the system for constructible falsity given in NELSON 1949 (and also in THOMASON 1969). However, the schema $\neg A \supset (A \supset B)$ is provable in Nelson's system but not in **RFC**. The simplest way to show that $\neg A \supset (A \supset B)$ is not **RFC**-provable is to give a Kripke-style semantics for **RFC**.

## § 3. AN ALTERNATIVE INTERPRETATION FOR RFC

The interpretation of **RFC** using constructions is the most natural one from a constructive viewpoint. However before one could even attempt to prove the completeness of **RFC** under such interpretation one would have to set up a precise calculus for constructions. Since the latter are usually rather complicated we prefer to suggest an alternative interpretation for **RFC** using a modification of the Kripke models for intuitionism.

In order to avoid having too many subscripts we shall assume that formal language has only one non-logical constant; a binary relation symbol $P$. Then by an **RFC**-*structure* we understand a system of the form:

$$\mathbf{K} = \langle \{\mathfrak{A}_i\}_{i \in I}, \leqslant \rangle$$

where

*$I$ is a non empty set,*

*$\leqslant$ is a reflexive, transitive relation on $I$,*

*$\mathfrak{A}_i = \langle A_i, R_i^+, R_i^- \rangle$ where $A_i$ is a non-empty (inhabited) set and $R_i^+, R_i^-$ are subsets of $A_i \times A_i$.*

*If $i \leqslant j$ then $A_i \subseteq A_j$, $R_i^+ \subseteq R_j^+$ and $R_i^- \subseteq R_j^-$.*

If $\mathbf{K} = \langle \{\mathfrak{A}_i\}_{i \in I}, \leqslant \rangle$ is an **RFC**-structure and $j \in I$, then $\langle \mathbf{K}, j \rangle$ (or simply $\mathbf{K}_j$) will be called an **RFC**-*realization* and $A_j$ the *universe* of $K_j$.

The way to interpret an **RFC**-structure $\mathbf{K} = \langle \{\mathfrak{A}_i\}_{i \in I}, \leqslant \rangle$ is as follows: The elements of the index $I$ are viewed as (possible) stages of knowledge and the elements of $A_i$ as the individuals obtained by stage $i$. $R_i^+$ consists of those pairs $(a, b)$ of elements of $A_i$ for which there is evidence that $P(a, b)$ holds; $R_i^-$ consists of those pairs of elements $(a, b)$ for which there is evidence that $P(a, b)$ does not hold. Since it is possible that at stage $i$ there is neither evidence that $P(a, b)$ holds nor that $P(a, b)$ does not hold, it is not natural to require that the union of $R_i^+$ and $R_i^-$ be $A_i \times A_i$. Also since at any given stage $i$ it is possible to have conflicting evidence, we must allow the possibility that the intersection of $R_i^+$ and $R_i^-$ be non-empty.

The formal definition of satisfaction is as follows:

DEFINITION 1. *If $\mathbf{K} = \langle \{\mathfrak{A}_j\}_{j \in I}, \leqslant \rangle$ is an* **RFC**-*structure, $i \in I$ and $s$ is a $\mathbf{K}_i$-assignment* (i.e. *a function from the individual variables into the universe of $\mathbf{K}_i$*) *then $\models_{\mathbf{K}_i} A[s]$ and $=\!|_{\mathbf{K}_i} A[s]$ are defined as follows: (we omit the symbol $\mathbf{K}$)*

$A = P(x, y)$ $\quad \models_i A[s] \quad$ *iff* $\langle s(x), s(y) \rangle \in R_i^+$

$\qquad\qquad\quad =\!|_i A[s] \quad$ *iff* $\langle s(x), s(y) \rangle \in R_i^-$

| | | |
|---|---|---|
| $A = B \mathbin{\&} C$ | $\models_i A[s]$ | *iff* $\models_i B[s]$ *and* $\models_i C[s]$ |
| | $=\!\!|_i A[s]$ | *iff* $=\!\!|_i B[s]$ *or* $=\!\!|_i C[s]$ |
| $A = B \vee C$ | $\models_i A[s]$ | *iff* $\models_i B[s]$ *or* $\models_i C[s]$ |
| | $=\!\!|_i A[s]$ | *iff* $=\!\!|_i B[s]$ *and* $=\!\!|_i C[s]$ |
| $A = B \supset C$ | $\models_i A[s]$ | *iff* $\models_j C[s]$ *whenever* $\models_j B[s]$ *and* $i \leqslant j$ |
| | $=\!\!|_i A[s]$ | *iff* $\models_i B[s]$ *and* $=\!\!|_i C[s]$ |
| $A = \exists x B(x)$ | $\models_i A[s]$ | *iff for some* $a \in A_i$, $\models_i B[s(x/a)]$ |
| | $=\!\!|_i A[s]$ | *iff for all* $a \in A_j$ *and* $j \geqslant i$, $=\!\!|_j B[s(x/a)]$ |
| $A = \forall x B(x)$ | $\models_i A[s]$ | *iff for* $j \geqslant i$ *and* $a \in A_j$, $\models_j B[s(x/a)]$ |
| | $=\!\!|_i A[s]$ | *iff for some* $a \in A_i$, $=\!\!|_i B[s(x/a)]$ |
| $A = \neg B$ | $\models_i A[s]$ | *iff* $=\!\!|_i B[s]$ |
| | $=\!\!|_i A[s]$ | *iff* $\models_i B[s]$. |

Definition 2. *A formula $A$ is* **RFC**-*valid, in symbols*: $\models A$, *iff for all* **RFC**-*realizations* $\mathbf{K}_i$ *and* $\mathbf{K}_i$-*assignments* $s$, $\models_{\mathbf{K}_i} A[s]$.

We extend the definition of satisfaction to sequents as follows:

Definition 3. *If $S$ is the sequent $A_0, \ldots, A_{k-1} \rightarrow B$, then* $\models_{\mathbf{K}_i} S$ $[=\!\!|_{\mathbf{K}_i} S]$ *iff* $\models_{\mathbf{K}_i}(A_0 \mathbin{\&} \ldots \mathbin{\&} A_{k-1} \supset B)$ $[=\!\!|_{\mathbf{K}_i}(A_0 \mathbin{\&} \ldots \mathbin{\&} A_{k-1} \supset B)]$.

The usual proof by induction on the length of the derivation gives that

Theorem 2. *If the formula $A$ is* **RFC**-*provable then it is* **RFC**-*valid.*

## § 4. A completeness theorem for RFC

A reasonable, but not conclusive test whether **RFC** is a correct formalization of the intuitive concepts behind the definition of validity using the **RFC**-structures would be a completeness theorem. If one is willing to use non-constructive means then the methods used by Aczel 1968 to prove the completeness of the intuitionistic predicate calculus can easily be adapted to give

Theorem 3. (*A completeness theorem for* **RFC**). *A formula $A$ is* **RFC**-*provable iff it is* **RFC**-*valid.*

It would be more interesting to obtain the completeness theorem by constructive means, but since **RFC** includes intuitionism (provided enough predicates are used, see next section) it is unlikely that a constructive proof could be found.

On the other hand, even though the proof of the completeness theorem may not be philosophically gratifying, at least the semantics introduced can be used to obtain simple proofs of the independence of the following schemata (and many others):

$$\neg A \supset (A \supset B),\ A \vee \neg A,\ (A \supset B) \supset ((A \supset \neg B) \supset \neg A),\ \neg (A \mathbin{\&} \neg A).$$

## § 5. An elementary number theory based on **RFC**

The natural numbers is the obvious place to try out the applicability of **RFC**. One way would be to add to **RFC** the standard Peano's axioms. However such approach would be arbitrary and would be no more than an exercise in axiomatics. A more satisfying method is to go back to the intuitive concept of a natural number and then decide which sentences (sequents) should be chosen as axioms.

We shall assume that a natural number is a construction, in fact we shall identify the natural number $n$ with the construction that starts by construction 0 and ends with the construction of $n$. Furthermore we accept Heyting's observation on page 13 of Heyting 1966 that:

*"The notion of natural number does not come to us as a bare notion, but from the beginning it is clothed in properties which I can detect by simple observation".*

To use the natural numbers come clothed with the following concepts: $\doteq$ (equality), $^+$ (successor), $^-$ (predecessor [1])), $+$ (addition), $\cdot$ (multiplication) and $\mathbf{0}$ (zero).

The following sequents are to be included in the axioms since they can all be positively justified:

$N_1$ $$a \doteq b \rightarrow a^+ \doteq b^+$$

$N_2$ $$a^+ \doteq b^+ \rightarrow a \doteq b$$

$N_3$ $$\rightarrow a+0 \doteq a$$

$N_4$ $$\rightarrow a+b^+ \doteq (a+b)^+$$

$N_5$ $$\rightarrow a\cdot 0 \doteq 0$$

$N_6$ $$\rightarrow a\cdot b^+ \doteq a\cdot b+a$$

$N_7$ $$\rightarrow 0^- \doteq 0$$

$N_8$ $$\rightarrow a^{+-} \doteq a$$

$N_9$ $$a \doteq b,\, a \doteq c \rightarrow b \doteq c.$$

The following schemas, although they may implicitly involve refutation must surely be included in any meaningful development of the natural numbers:

$N_{10}$ $$A(0),\, \forall x(A(x) \supset A(x^+)) \rightarrow A(x)$$

$N_{11}$ $$t \doteq s,\, A(t) \rightarrow A(s).$$

Of the axioms which explicitly involve $\neg$ the one hardest to justify is probably:

$N_{12}$ $$\rightarrow \neg\, 1 \doteq 0$$

[1]) The use of the predecessor function simplifies the exposition.

where $\mathbf{1}=0^{+}$ (and in general: $\mathbf{n}=0^{+\cdots+}$). Arguments in its favour can be found in BROUWER 1954. If one accepts $N_{12}$ then the following, which basically involve changing a refutation having a particular property into a refutation with some other property, should cause no problem:

$N_{13}$ $\quad \neg\, a \doteq b \rightarrow \neg\, b \doteq a$

$N_{14}$ $\quad \neg\, a \doteq b \rightarrow \neg\, b^{+} \doteq a^{+}$

$N_{15}$ $\quad \neg\, a^{+} \doteq b^{+} \rightarrow \neg\, a \doteq b$

$N_{16}$ $\quad \neg\, a \doteq 0 \rightarrow \neg\, a^{+} \doteq 0.$

And finally the following axioms, although more problematic than $N_{13}$–$N_{16}$ because they involve going from refutations to proofs and conversely, are nevertheless no more awkward than $N_{12}$:

$N_{17}$ $\quad \neg\, a \doteq 0 \rightarrow a^{-+} \doteq a$

$N_{18}$ $\quad a^{-+} \doteq a \rightarrow \neg\, a \doteq 0.$

By **PN** we understand the formal system of elementary number theory whose logical basis is **RFC** and whose number theoretic postulates are $N_1$–$N_{18}$,

It is clear that **PN** is a subtheory of classical number theory, furthermore it is a proper subtheory of classical number theory because it is a subtheory of the system $\mathbf{N_1}$ for constructible falsity introduced by NELSON 1949 (where it is also shown, by the use of realizability conditions that $\mathbf{N_1}$ is a proper subtheory of classical number theory).

The basic difference between **PN** and Nelson $\mathbf{N_1}$ is that in the underlying logic for $\mathbf{N_1}$ the law of the denial of the antecedent, $\neg\, A \supset (A \supset B)$, is accepted as an axiom while in **RFC** it is not. The reason for not including it in **RFC** (or rather its equivalent sequent formulation) is that it is difficult to visualize how to obtain, for arbitrary formulae $A$ and $B$, a construction $q$ which proves $B$ from constructions $p$, $r$ such that $p$ proves $A$ and $r$ refutes $A$; specially in the case when $B$ has nothing to do with $A$! Of course if $A$ and $B$ are both number theoretic formulae then the rejection of $\neg\, A \supset (A \supset B)$ on the grounds that $A$ and $B$ may have nothing to do with each other is questionable.

It turns out that in the case of **PN** the question of whether the law of the denial of the antecedent should be included or does not come up because it is provable in **PN**. In other words, **RFC** plus the axioms $N_1$–$N_{18}$ (which are acceptable on grounds consistent with the principles of **RFC**) allows us to derive the schema $\neg\, A \supset (A \supset B)$ and thus **PN** is (extensionally) equivalent to Nelson's system $\mathbf{N_1}$.

The derivation of $\neg\, A \supset (A \supset B)$ in **PN** is best broken down into two steps. In the first it is shown that $\mathbf{1} \doteq 0 \supset B$ and in the second that $A \mathbin{\&} \neg\, A \supset \mathbf{1} \doteq 0$.

LEMMA 1. *The schema* $1 \doteq 0 \rightarrow B$ *is provable in* **PN**.

PROOF. By repeated application of the axioms (specially the induction schema $N_{10}$) it can be shown that the following sequents are provable in **PN**

$$a^+ \doteq 0 \rightarrow 1 \doteq 0$$

$$\rightarrow \neg\, a^+ \doteq 0$$

$$1 \doteq 0 \rightarrow a \doteq 0$$

$$(*) \qquad 1 \doteq 0 \rightarrow a \doteq b$$

$$1 \doteq 0 \rightarrow \neg\, a \doteq 0$$

$$(**) \qquad 1 \doteq 0 \rightarrow \neg\, a \doteq b$$

Then starting with (*) and (**) one can prove, by induction on the complexity of the formula $B$, that $1 \doteq 0 \rightarrow B$ is provable in **PN**.

LEMMA 2. *The schema* $A, \neg A \rightarrow 1 \doteq 0$ *is provable in* **PN**.

Proof is by induction on the complexity of $A$.

*Basis step.* $A$ is an atomic formula. First observe that from $N_{17}$ we obtain $\vdash_{PN} \neg\, 0 \doteq 0 \rightarrow 1 \doteq 0$ from which it follows that $\vdash_{PN} \rightarrow \neg\, 0 \doteq 0 \supset 1 \doteq 0$. The latter can then be used as the basis step in the proof (via $N_{10}$) of $\vdash_{PN} \rightarrow \neg\, a \doteq a \supset 1 \doteq 0$. Then using $N_{11}$ we obtain $\vdash_{PN} a \doteq b, \neg\, a \doteq b \rightarrow 1 \doteq 0$.

*Induction step.* We shall consider one case, namely when $A = \exists x B(x)$; the others are similar. From the induction hypothesis we have that $\vdash_{PN} \rightarrow B(x) \;\&\; \neg\, B(x) \supset 1 \doteq 0$. Using the quantifier rules we can then obtain $\vdash_{PN} \rightarrow \forall x \exists y (B(x) \;\&\; \neg\, B(y) \supset 1 \doteq 0)$, which in turn leads to $\vdash_{PN} \rightarrow \exists x B(x) \;\&\; \forall y \neg\, B(y) \supset 1 \doteq 0$. Then using the equivalences (3) of theorem 1 we obtain $\vdash_{PN} \rightarrow \exists x B(x) \;\&\; \neg\, \exists y B(y) \supset 1 \doteq 0$. The latter is then used in order to show that $\vdash_{PN} \exists x B(x), \neg\, \exists x B(x) \rightarrow 1 \doteq 0$.

Combining the two lemmas we obtain:

THEOREM 4. *The system* **PN** *is equivalent to Nelson's system* $\mathbf{N_1}$.

It trivially follows that the results obtained by Nelson for $\mathbf{N_1}$ can now be applied to **PN**, e.g. translations into intuitionistic systems. An application that can be made of the translations into the intuitionistic system is to show that if $A$, $B$ and $\exists x C(x)$ are sentences then

(a) $\vdash_{PN} A \vee B$ iff either $\vdash_{PN} A$ or $\vdash_{PN} B$

(b) $\vdash_{PN} \exists x C(x)$ iff for some natural number $n$, $\vdash_{PN} C(\mathbf{n})$.

## § 6. INFINITARY EXTENSIONS OF PN (AND $N_1$)

Since it can be argued that it is easier to visualize certain infinitudes than some negations it may be worthwhile to extend **PN** (or $\mathbf{N_1}$) by allowing infinitary rules of inference; for example Carnap's rule:

($\omega$-rule) *To conclude* $\Gamma \rightarrow \forall x A(x)$ *from* $\Gamma \rightarrow A(\mathbf{0}), \ldots, \Gamma \rightarrow A(\mathbf{n}), \ldots$

The first observation that can be made about adding the unrestricted $\omega$-rule to **PN** is

THEOREM 5. *If* $\mathbf{PN}_\omega$ *is the system obtained by adding the unrestricted* $\omega$*-rule to* **PN** *then a sentence* $A$ *is provable in* $\mathbf{PN}_\omega$ *iff* $A$ *is (classically) true.*

PROOF. It can be shown by induction on the complexity of $A$ [1]) that: If $A$ is true then $\vdash_{\mathbf{PN}_\omega} \rightarrow A$ and if $A$ is false then $\vdash_{\mathbf{PN}_\omega} \rightarrow \neg A$.

From a constructive viewpoint it is more natural to add a restricted $\omega$-rule. For example one could require that there be a recursive function $\varphi$ such that for each natural number $n$, $\varphi(n)$ is a Godel number of a *proof* of $\Gamma \rightarrow A(\mathbf{n})$ in order to be allowed to derive $\Gamma \rightarrow \forall x A(x)$ from $\Gamma \rightarrow A(\mathbf{0}), \ldots, \Gamma \rightarrow A(\mathbf{n}), \ldots$.

Let $\mathbf{PN}_{\omega-\text{rec}}$ be the system obtained by adding the recursively restricted $\omega$-rule to **PN**.

The notion of $P$-realizability introduced in NELSON 1949 can be extended to sequents and then applying the fixed point theorem for partial recursive functions one obtains:

THEOREM 6. *If a sentence* $A$ *is provable in* $\mathbf{PN}_{\omega-\text{rec}}$ *then* $A$ *is* $P$*-realizable.*

From theorem 6 it follows that (a) not all classically true formulae are provable in $\mathbf{PN}_{\omega-\text{rec}}$ and (b) that some intuitionistic acceptable formulae are not provable in $\mathbf{PN}_{\omega-\text{rec}}$.

On the other hand it can be shown that if $A''$ is the formula obtained from $A$ by replacing all parts of the form $\neg B$ by $(B \supset \mathbf{1} \doteq \mathbf{0})$ then $A''$ is provable in $\mathbf{PN}_{\omega-\text{rec}}$ iff $A$ is provable in $\mathbf{HA}_{\omega-\text{rec}}$, intuitionistic arithmetic with the recursively restricted $\omega$-rule (c.f. with theorem 4 of Nelson 1949). Also if $A^*$ is the Godel translation of $A$ then it can be shown that $A^*$ is provable in $\mathbf{HA}_{\omega-\text{rec}}$ iff $A$ is provable in $\mathbf{PA}_{\omega-\text{rec}}$. [2]) But by a result of SHOENFIELD 1959, a sentence is provable in $\mathbf{PA}_{\omega-\text{rec}}$ iff it is classically true. Thus we have

THEOREM 7. *A sentence* $A$ *of elementary number theory is classically true iff* $(A^*)''$ *is provable in* $\mathbf{PN}_{\omega-\text{rec}}$.

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[1]) The most tedious case is when $A$ is atomic.

[2]) By $\mathbf{PA}_{\omega-\text{rec}}$ we understand classical number theory with the recursively restricted $\omega$-rule.

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