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SENTENTIAL CALCULUS FOR LOGICAL FALSEHOODS

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Several axiomatic systems for sentential calculus have been developed. Such systems are generally motivated by a consideration of logically true sentences of the formal language. In this paper I present a finitely axiomatized system of sentential calculus for logically false sentences.

1. *Introduction*. Consider a formal language L with the following symbols:

Sentential variables: $P_1, P_2, \ldots$
Sentential connectives: &—"and," v—"or," $\neg$—"not"
Punctuation:), and (

I will assume the standard definition for "sentence of L." The meta-symbols $R, R_1, R_2, \ldots$ will be used to refer to sentences of L. In addition, I will presuppose the standard theory of two-valued truth tables. I will say that a sentence of L is logically true (**LT**) if and only if the final column of its truth table has only **T**'s. I will say that a sentence of L is logically false (**LF**) if and only if the final column of its truth table has only **F**'s. I will say that two sentences R_1 and R_2 of L are logically equivalent (R_1 **LE** R_2) if and only if the sentence $(R_1 \& R_2) \vee (\neg R_1 \& \neg R_2)$ is **LT**.

2. *The System* **SCT**. In [1], Hilbert and Ackermann present an axiomatic system of sentential calculus for logical truths. With some small notational differences, their system uses the symbols mentioned above and in addition the symbol "$\rightarrow$". As they note, however, this symbol is to be considered an abbreviation; if R_1 and R_2 are any two sentences of the language, then $R_1 \rightarrow R_2$ is to be considered an abbreviation for the sentence $\neg R_1 \vee R_2$ ([1], pp. 27-28). In discussing their system, I will eliminate this abbreviation. Since their system is primarily concerned with **LT** sentences, I will refer to their system as **SCT** (sentential calculus for truths). With slight notational differences and the removal of the symbol "$\rightarrow$", the Hilbert and Ackermann system may be presented as follows:

Axioms:

(ta) $\neg(P_1 \vee P_1) \vee P_1$

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(tb) $\neg P_1 \vee (P_1 \vee P_2)$
(tc) $\neg(P_1 \vee P_2) \vee (P_2 \vee P_1)$
(td) $\neg(\neg P_1 \vee P_2) \vee (\neg(P_3 \vee P_1) \vee (P_3 \vee P_2))$

Rules of Proof:

1. Rule of substitution: We may substitute in a given sentence of L for a sentential variable any sentence of L, providing that the substitution is made wherever that sentential variable occurs.
2. Rule of T-implication: From two sentences R_1 and $\neg R_1 \vee R_2$ the sentence R_2 may be obtained.

I will assume the standard definition of "proof in **SCT**." If the sentence R of L is provable in **SCT**, I will write $\vdash_T R$. Let λ be a set of sentences of L. If R is provable in **SCT** from the axioms augmented by the members of λ, I will write $\lambda \vdash_T R$. Using these notions, the following theorems may be proved about **SCT** ([1], chapter 1):

Theorem A: *If R is any sentence of L and $\vdash_T R$, then R is* **LT**.
Theorem B: **SCT** *is consistent in the sense that there is no sentence R of L such that both $\vdash_T R$ and $\vdash_T \neg R$.*
Theorem C: *The axioms* ta-td *are independent. That is, it is not possible to prove any one of the axioms from all of the others.*
Theorem D: *The system* **SCT** *is complete in the sense that if R is any sentence of L that is* **LT**, *then $\vdash_T R$.*
Theorem E: *The system* **SCT** *is complete in the sense that if a sentence of L that is not provable from the axioms is added to the system as an axiom, the new system is inconsistent.*

3. *The System* **SCF**. I will now present a system of sentential calculus for **LF** sentences of L; I will refer to this system as **SCF**. I will prove theorems about **SCF** that are analagous to the theorems presented above about **SCT**.

Axioms:

(fa) $\neg(P_1 \& P_1) \& P_1$
(fb) $\neg P_1 \& (P_1 \& P_2)$
(fc) $\neg(P_1 \& P_2) \& (P_2 \& P_1)$
(fd) $\neg(\neg P_1 \& P_2) \& (\neg(P_3 \& P_1) \& (P_3 \& P_2))$

Rules of Proof:

1. Rule of substitution: same as that for **SCT**.
2. Rule of F-implication: From the two sentences R_1 and $\neg R_1 \& R_2$ the sentence R_2 may be obtained.

I will again assume the standard definition of "proof in **SCF**." If the sentence R of L is provable in **SCF**, I will write $\vdash_F R$. As for **SCT**, if λ is a set of sentences of L, I will write $\lambda \vdash_F R$ when R is provable in **SCF** from the axioms augmented by the members of λ. Theorems analogous to those presented above for **SCT** could be proven in a straightforward manner for

SCF without reference to **SCT**. However, I will construct proofs for such theorems (except Theorem A′) by relating them to the theorems for **SCT**.

In the following material, I will make use of the one-to-one function F from the set of sentences of L onto the set of sentences of L, defined in the following way:

(a) $F(P_i) = P_i$, for any sentential variable P_i
(b) $F(\neg R) = \neg F(R)$, for any sentence R of L
(c) $F(R_1 \vee R_2) = F(R_1)\ \&\ F(R_2)$, for any sentences R_1 and R_2 of L
(d) $F(R_1\ \&\ R_2) = F(R_1) \vee F(R_2)$, for any sentences R_1 and R_2 of L

Theorem 1: *Let R be a sentence of L. Then $F(F(R)) = R$.*

Proof: By induction on the number, n, of connectives in R; n will be called the length of R. Suppose $n = 0$. Then R is a sentential variable, say P_i. Then by (a), $F(F(P_i)) = F(P_i) = P_i$. Suppose the theorem is true for all n less than some number p, p greater than 0. We must show that the theorem holds for p. Let R be an arbitrary sentence of length p. Then R must be of the form $\neg R_1$, $R_1 \vee R_2$, or $R_1\ \&\ R_2$. Suppose for some R_1, $R = \neg R_1$. Then by (b), $F(F(R)) = F(F(\neg R_1)) = F(\neg F(R_1)) = \neg F(F(R_1))$. But R_1 has length $p - 1$, and thus by assumption the theorem holds for R_1. Thus $F(F(R_1)) = R_1$. Thus $F(F(R)) = \neg R_1 = R$. Similarly it is easy to show that if R has the form $R_1 \vee R_2$ or $R_1\ \&\ R_2$ then the theorem holds. Thus the theorem holds for $n = p$, and hence for all n. The following lemmas will be needed for the proof of the next theorem.

Lemma 1: *If R_1 is the result of substituting R_2 in R_3 for the sentential variable P_i, then $F(R_1)$ is the result of substituting $F(R_2)$ in $F(R_3)$ for P_i.*

Proof: By induction on the length, n, of R_3. Suppose $n = 0$. Then R_3 must be a sentential variable. Suppose R_3 is P_i. Then $R_1 = R_2$, and $F(R_1) = F(R_2)$. But $F(R_3) = P_i$. Thus the result of substituting $F(R_2)$ for P_i in $F(R_3)$ is just $F(R_2)$. Hence $F(R_1)$ is the result of substituting $F(R_2)$ in $F(R_3)$ for P_i. Suppose R_3 is P_j, where $P_j \neq P_i$. Then $R_1 = R_3$, and thus R_1 is P_j. But then both $F(R_1)$ and $F(R_3)$ are just P_j. Hence, $F(R_1)$ is the result of substituting $F(R_2)$ in $F(R_3)$ for P_i. Thus the lemma holds for $n = 0$. Now, suppose the lemma is true for all n less than some number p, where p is greater than 0. Let R_3 be an arbitrary sentence of L of length p.

Case 1: $R_3 = \neg R_4$, for some R_4. Then $R_1 = \neg R_5$, for some R_5, where R_5 is the result of substituting R_2 in R_4 for the sentential variable P_i. But R_4 is of length $p - 1$. Hence by induction hypothesis, $F(R_5)$ is the result of substituting $F(R_2)$ in $F(R_4)$ for the sentential variable P_i. But $F(R_3)$ is just $F(\neg R_4)$ which is $\neg F(R_4)$; further, $F(R_1)$ is just $F(\neg R_5)$ which is $\neg F(R_5)$. Thus $F(R_1)$ is the result of substituting $F(R_2)$ in $F(R_3)$ for P_i.

Case 2: $R_3 = R_4 \vee R_5$, for some R_4 and R_5. Then $R_1 = R_6 \vee R_7$, where R_6 is the result of substituting R_2 in R_4 for P_i and R_7 is the result of substituting R_2 in R_5 for P_i. Since the length of R_4 is less than p and the length of R_5 is less than p, we have $F(R_6)$ is the result of substituting $F(R_2)$ in $F(R_4)$ for P_i and $F(R_7)$ is the result of substituting $F(R_2)$ in $F(R_5)$ for P_i.

But $F(R_1)$ is just $F(R_6 \vee R_7)$ which is $F(R_6)$ & $F(R_7)$; further, $F(R_3)$ is just $F(R_4 \vee R_5)$ which is $F(R_4)$ & $F(R_5)$. Thus $F(R_1)$ is the result of substituting $F(R_2)$ in $F(R_3)$ for P_i.

Case 3: $R_3 = R_4$ & R_5, for some R_4 and R_5. This case is exactly the same as Case 2, interchanging "&" and "v".

Thus the lemma is true for $n = p$, and hence it is true for all n.

Lemma 2: *If $F(R_1)$ is the result of substituting $F(R_2)$ in $F(R_3)$ for the sentential variable P_i, then R_1 is the result of substituting R_2 in R_3 for P_i.*

Proof: Suppose the hypothesis of the lemma is true. Then by Lemma 1, $F(F(R_1))$ is the result of substituting $F(F(R_2))$ in $F(F(R_3))$ for P_i. But then by Theorem 1, R_1 is the result of substituting R_2 in R_3 for P_i.

The next theorem will be of fundamental importance in the work to follow. It simply tells us that if we have a proof in either **SCF** or **SCT**, then we can transform it into a proof in the other system by means of the function F.

Theorem of Proof Correspondence: *Let $R_1, R_2, \ldots, R_n$ be a series of sentences of L, and let $F(R_1), F(R_2), \ldots, F(R_n)$ be a series of sentences of L obtained from the first by taking the F-transformation of each sentence in that series. Then $R_1, R_2, \ldots, R_n$ constitutes a proof of R_n in* **SCT** *if and only if $F(R_1), F(R_2), \ldots, F(R_n)$ constitutes a proof of $F(R_n)$ in* **SCF**.

Proof: By induction on the number, n, of steps in the proof; n will be called the length of the proof. The theorem may be broken into two parts.

First suppose $R_1, R_2, \ldots, R_n$ is a proof in **SCT** of R_n. We want to show that $F(R_1), F(R_2), \ldots, F(R_n)$ is a proof of $F(R_n)$ in **SCF**. Suppose $n = 1$. Then R_1 is an axiom of **SCT**. But $fa = F(ta), \ldots$, and $fd = F(td)$. Hence $F(R_1)$ is a proof of $F(R_1)$ in **SCF**, and thus this half of the theorem holds for $n = 1$. Now, suppose this first half of the theorem holds for all n less than some number p, where p is greater than 1. We must show that this half of the theorem holds for p. Suppose $R_1, R_2, \ldots, R_p$ is a proof in **SCT** of R_p. We want to show that $F(R_1), F(R_2), \ldots, F(R_p)$ is a proof in **SCF** of $F(R_p)$. The only question that may arise concerns step p, for by the induction hypothesis, the steps through $p - 1$ constitute a proof. Consider the justification for step p.

Case 1: R_p is an axiom. This case is the same as for $n = 1$.

Case 2: R_p follows by substitution of the sentence R in R_j for the sentential variable P_i. Then by Lemma 1, $F(R_p)$ follows by substitution of the sentence $F(R)$ in $F(R_j)$ for the sentential variable P_i.

Case 3: R_p follows by T-implication from two previous sentences R_i and R_j, where $R_j = \neg R_i \vee R_p$. But then $F(R_j)$ is just $F(\neg R_i \vee R_p)$ which is $\neg F(R_i)$ & $F(R_p)$. Hence $F(R_p)$ follows from $F(R_i)$ and $F(R_j)$ by F-implication.

Thus the first half of the theorem holds for $n = p$, and hence for all n. The proof of the second half of the theorem relies on the fact that $F(F(R)) = R$ (Theorem 1) and is exactly similar to the above proof, using Lemma 2 in place of Lemma 1.

Corollary 1: $\vdash_T R$ *if and only if* $\vdash_F F(R)$.
Corollary 2: *Let* λ *be a set of sentences, and let* $F(\lambda)$ *be the set of sentences whose members are the* F*-transforms of the members of* λ. *Then* $\lambda \vdash_T R$ *if and only if* $F(\lambda) \vdash_F F(R)$.

Proof: Note that a step R_i in the **SCT** proof is a member of λ if and only if $F(R_i)$ is a member of $F(\lambda)$. The proof is then the same as the proof for the Theorem of Proof Correspondence.

I will now proceed to prove theorems analogous to the first three presented for **SCT**, above.

Theorem A′: *If* R *is any sentence of* L *and* $\vdash_F R$, *then* R *is* **LF**.

Proof: The proof is exactly parallel to the proof for Theorem A. Note that all of the axioms for **SCF** are **LF**, and that the rules of proof preserve the property of being **LF**. The theorem then follows immediately.

Theorem B′: **SCF** *is consistent in the sense that there is no sentence* R *of* L *such that* $\vdash_F R$ *and* $\vdash_F \neg R$.

Proof: Suppose there were a sentence R such that $\vdash_F R$ and $\vdash_F \neg R$. Then by Theorem 1, $\vdash_F F(F(R))$ and $\vdash_F \neg F(F(R))$. Then by definition of F, $\vdash_F F(\neg F(R))$. Then by Corollary 1, $\vdash_T F(R)$ and $\vdash_T \neg F(R)$. But this contradicts Theorem B. Hence there is no sentence R of L such that $\vdash_F R$ and $\vdash_F \neg R$.

Theorem C′: *The axioms fa-fd are independent. That is, it is not possible to prove any one of the axioms from all of the others.*

Proof: Suppose the theorem were false, and that there is a proof of one of the axioms from the others. Note that $ta = F(fa)$, . . . , and $td = F(fd)$. Then by the Theorem of Proof Correspondence, one could transform the **SCF** proof into an **SCT** proof in which one of the **SCT** axioms is proven from the others. But this contradicts Theorem C. Hence the axioms *fa-fd* are independent.

I will prove a few intermediate theorems before proving the remaining analogous theorems.

Theorem 2: *For any sentence* R *of* L, *if* R *is* **LT**, *then* $F(R)$ *is* **LF**.

Proof: Suppose R is **LT**. Then by Theorem D, $\vdash_T R$. Then by Corollary 1, $\vdash_F F(R)$. Then by Theorem A′, $F(R)$ is **LF**.

Theorem 3: *If* $F(R)$ *is* **LF**, *then* R *is* **LT**, *for* R *an arbitrary sentence of* L.

Proof: Suppose $F(R)$ is **LF**. Then $\neg F(R)$ is **LT**. Thus $F(\neg R)$ is **LT**. Then by Theorem 2, $F(F(\neg R))$ is **LF**. By Theorem 1, $\neg R$ is **LF**. Hence R is **LT**.

Theorem 4: *Let* R, R_1, *and* R_2 *be sentences of* L. *The following are equivalent:*

(*a*) R *is* **LT** *if and only if* $F(R)$ *is* **LF**,
(*b*) R *is* **LF** *if and only if* $F(R)$ *is* **LT**,
(*c*) R_1 **LE** R_2 *if and only if* $F(R_1)$ **LE** $F(R_2)$.

Proof: (a) implies (b): Suppose (a) is true. Let R be an arbitrary sentence of L that is **LF**. Then $\neg R$ is **LT**. By (a), $F(\neg R)$ is then **LF**, and hence $\neg F(R)$ is **LF**. Thus $F(R)$ is **LT**. Let R be an arbitrary sentence of L such that $F(R)$ is **LT**. Then by (a), $F(F(R))$ is **LF**. By Theorem 1, R is **LF**.

(b) implies (c): Suppose (b) is true. Let R_1 and R_2 be arbitrary sentences of L such that R_1 **LE** R_2. Then $(R_1 \,\&\, R_2) \vee (\neg R_1 \,\&\, \neg R_2)$ is **LT**. Thus $F(F((R_1 \,\&\, R_2) \vee (\neg R_1 \,\&\, \neg R_2)))$ is **LT**, by Theorem 1. By (b), $F((R_1 \,\&\, R_2) \vee (\neg R_1 \,\&\, \neg R_2))$ is **LF**, and thus $\neg F((R_1 \,\&\, R_2) \vee (\neg R_1 \,\&\, \neg R_2))$ is **LT**. But then $(F(R_1) \,\&\, F(R_2)) \vee (\neg F(R_1) \,\&\, \neg F(R_2))$ is **LT**. Hence, $F(R_1)$ **LE** $F(R_2)$. To prove R_1 **LE** R_2 if $F(R_1)$ **LE** $F(R_2)$, it is only necessary to reverse the steps in the above argument.

(c) implies (a): Suppose (c) is true. Assume R is **LT**. Then R **LE** $P_1 \vee \neg P_1$. Thus by (c), $F(R)$ **LE** $F(P_1 \vee \neg P_1)$. But $F(P_1 \vee \neg P_1)$ is just $P_1 \,\&\, \neg P_1$, which is **LF**. Thus $F(R)$ is **LF**. Now, assume $F(R)$ is **LF**. Then $F(R)$ **LE** $P_1 \,\&\, \neg P_1$. By (c), $F(F(R))$ **LE** $F(P_1 \,\&\, \neg P_1)$. But $F(F(R)) = R$ by Theorem 1, and $F(P_1 \,\&\, \neg P_1) = P_1 \vee \neg P_1$. Thus R **LE** $P_1 \vee \neg P_1$. Hence R is **LT**.

Theorem 5: *R is* **LF** *if and only if $F(R)$ is* **LT**, *for any sentence R of L.*

Proof: The theorem follows directly from Theorems 2, 3, and 4.

Theorem 6: *R_1* **LE** *R_2 if and only if $F(R_1)$* **LE** *$F(R_2)$, for any sentences R_1 and R_2 of L.*

Proof: The theorem follows directly from Theorems 2, 3, and 4.

I will now prove the remaining two analogue theorems.

Theorem D′: **SCF** *is complete in the sense that if a sentence R of L is* **LF**, *then $\vdash_F R$.*

Proof: Suppose R is a sentence of L that is **LF**. Then by Theorem 5, $F(R)$ is **LT**. Thus by Theorem D, $\vdash_T F(R)$. By Corollary 1, $\vdash_F F(F(R))$. By Theorem 1, $\vdash_F R$.

Theorem E′: **SCF** *is complete in the sense that if a sentence of L that is not provable from the axioms is added to the system as an axiom, the new system will be inconsistent.*

Proof: Let R be a sentence of L such that not $\vdash_F R$. Then R is not **LF**, by Theorem D′. By Theorem 5, $F(R)$ is not **LT**. Add $F(R)$ to **SCT** as an axiom. By Theorem E, the new system is inconsistent. That is, for some sentence R_1 of L, $\{F(R)\} \vdash_T R_1$ and $\{F(R)\} \vdash_T \neg R_1$. By Corollary 2, $\{F(F(R))\} \vdash_F F(R_1)$ and $\{F(F(R))\} \vdash_F F(\neg R_1)$. But since $F(\neg R_1) = \neg F(R_1)$, $\{F(F(R))\} \vdash_F \neg F(R_1)$. By Theorem 1, $\{R\} \vdash_F F(R_1)$ and $\{R\} \vdash_F \neg F(R_1)$. Thus the new system is inconsistent.

It seems then that the system **SCF** has all of the "nice" logical properties that the system **SCT** possesses.

4. *Further Comments.* The obvious next step to take is to formulate a system of predicate calculus for logically false sentences. This work is currently in progress.

Another interesting problem that arises in connection with this study is developing an analogous finite axiomatization for logically contingent (**LC**) sentences. There are many difficulties. For example, consider possible rules of proof for such a system. Clearly the rule of substitution cannot be used as it stands. $P_1 \vee P_2$ is **LC**, but the result of substituting $\neg P_1$ for P_2 in that sentence would not be **LC**. This difficulty suggests a rule of substitution something like the following: We may substitute in a sentence of L for a sentential variable any **LC** sentence of L that has no sentential variables in common with the original sentence except perhaps the variable for which substitution is being made.

Other difficulties with rules of proof arise when one considers what possible deductions could be made from R_1 & R_2 or $R_1 \vee R_2$. If all we know about R_1 is that it is **LC** (has occurred in the proof), then we can conclude nothing about R_2 from either the fact that R_1 & R_2 is **LC** (has occurred in the proof) or the fact that $R_1 \vee R_2$ is **LC** (has occurred in the proof). This does suggest a rule something like: From R_1 & R_2 and $R_1 \vee R_2$, both R_1 and R_2 may be obtained.

Some oddities also arise in the meta-characterization of such a system. We would not want **SCC** to be consistent in the sense that there is no sentence such that both it and its negation are provable in the system. On the contrary, if a sentence is provable, then we want its negation to be provable as well, for if R is **LC**, then so is $\neg R$. Thus if the system is complete in the sense that for any **LC** sentence R of L, R is provable in **SCC**, then the system must be inconsistent in the above sense. However, we would want **SCC** to be consistent in the sense that not every sentence of L is provable in the system.

There are several points that lead me to suspect that **SCC** is finitely axiomatizable, in spite of these and other difficulties. First, of course, is the fact that **SCT** and **SCF** are both finitely axiomatizable. Secondly, the set of **LC** sentences of L is completely decidable by truth tables. It would seem odd to have a completely decidable set of sentences that was not finitely axiomatizable.

REFERENCE

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