REFUTATIONS IN LOGICS RELATED TO JOHANSSON'S, NELSON'S, AND SEGERBERG'S

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REFUTATION TAKEN SERIOUSLY

(*Indirect approach*) Refutation by failure to find a proof.

Refutations are not first-class citizens.

(*Direct approach*) Refutation by a single derivation.

Refutations are first-class citizens.

How about a pair?

$$(\mid\!-, -\!\mid)$$

|- searches for a proof, and -| searches for a refutation.

The logic **Int**

FOR is the set of all formulas generated from $VAR \cup \{\bot\}$ by $\wedge, \vee, \to$.
($\neg A = A \to \bot$ and $A \equiv B = (A \to B) \wedge (B \to A)$.)

Rules: *substitution* and *modus ponens*
Axioms:

$p \to (q \to p)$
$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$

$p \wedge q \to p \qquad p \wedge q \to q$
$(p \to q) \to ((p \to r) \to (p \to q \wedge r))$

$p \to p \vee q \qquad q \to p \vee q$
$(p \to r) \to ((q \to r) \to (p \vee q \to r))$

$\bot \to p$

The logic **J**

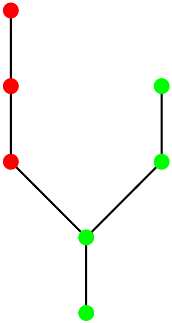
J is obtained from **Int** by removing (rejecting) the axiom

(I) $\bot \to p$
(the principle of explosion).

I. Johansson, Der Minimalkalkül, ein reduzierter intuitionistscher Formalismus, 1937.

J models

J frames $\mathcal{W} = (W, \leq, Q)$



J models $(\mathcal{W}, V)$

Mints-style normal forms

Definition

A *Mints-style normal form* (or a *general form*) is a formula

$$F = \Delta, \Gamma \longrightarrow a$$

where Δ is a finite set of formulas of the kind
$(b \to c) \to d$
and Γ is a finite set of formulas of the kind
b or $b \to C$ (C is either c or $c \to d$ or $c \vee d$).

(Here a, b, c, d are atoms.)

The *rank* of F is the number k of the formulas in Δ.
(We assume that $\Delta = \{(b_i \to c_i) \to d_i : 1 \leq i \leq k\}$.)

History

M. Wajsberg, Untersuchungen über den aussagenkalkül von A. Heyting, 1939.

G. Mints, Gentzen-type systems and resolution rules, LNCS, 1990.

J forms

Definition

A **J** *normal form* is a general form statisfying the following condition.
$(\star)$ If $b \to C \in \Gamma$ then $b \notin \Gamma$.

Lemma

For every formula A there is a **J** *normal form* F_A *such that* $\vdash A$ *iff* $\vdash F_A$.
(In fact, $\vdash A \to F_A$ *(by replacement).)*

F_A

Every subformula B of A has its "name" p_B.
(If B is an atom, then $p_B = B$.)

$F_A = \Sigma_A \longrightarrow p_A$
where Σ_A consists of equivalences
$(p_C \to p_D) \equiv p_B$, where $B = C \to D$ is a subformula of A.

$((p_C \wedge p_D) \equiv p_B$ is replaced by
$(p_C \to (p_D \to p_B), p_B \to p_C, p_B \to p_D$;
and $(p_C \vee p_D) \equiv p_B$ is replaced by
$(p_C \to p_B, p_D \to p_B, p_B \to p_C \vee p_D.)$

J forms

Definition

Let F be a **J** normal form.
(i) F is *abnormal* (or Q) iff $\bot \in \Gamma$.
(ii) F is *normal* (or non-Q) iff $\bot \notin \Gamma$.

Refutation system

(REFUTATION AXIOMS, REFUTATION RULES)

For example:

(*reverse substitution*) B/A where B is a substitution instance of A

(*reverse modus ponens*) B/A where $\vdash A \rightarrow B$

J. Łukasiewicz, *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, 1951.

J refutation axioms

Every special **J** normal form of rank 0.

Here by a *special* **J** *normal form* we mean a **J** normal form F such that $a \notin \Gamma$.

J refutation rules

$$(R) \qquad \frac{F_1, ..., F_k}{F}$$

where $F = \Sigma \longrightarrow a$ is a special **J** normal form of rank > 0 $(\Sigma = \Delta \cup \Gamma)$ and

$$F_i = \Sigma_i, b_i \longrightarrow c_i \qquad (1 \leq i \leq k).$$

$(\Sigma_i$ results from Σ by replacing $(b_i \to c_i) \to d_i$ with $c_i \to d_i$.)

Remark Since F_i is **J** equivalent to $\Sigma, b_i \longrightarrow c_i$, informally we also say that F_i is $\Sigma, b_i \longrightarrow c_i$ but Σ is reduced.

J refutation rules

(*Reverse Modus Ponens*) $\dfrac{A \to C}{B \to C}$ (where $\vdash A \to B$)

Note that $\vdash (B \to C) \to (A \to C)$.

RMP in fact

$$(R_i) \qquad \frac{G_i}{F}$$

where F is a special **J** normal normal form of rank > 0,
$G_i = \Sigma, (b_i \to c_i) \longrightarrow a \; (1 \le i \le k)$ and Σ is reduced.

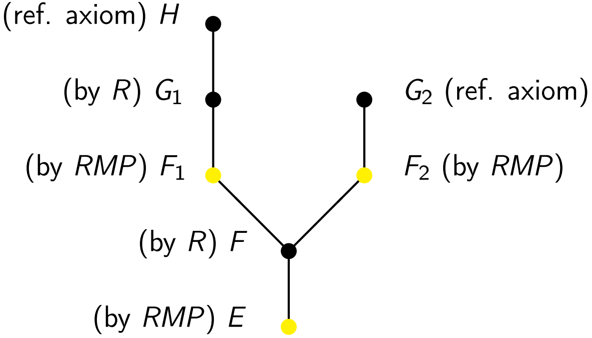
Normalization rules

$$\frac{A, B, \Psi \longrightarrow C}{A, A \to B, \Psi \longrightarrow C}$$

$$\frac{A, \Psi \longrightarrow C}{A \vee B, \Psi \longrightarrow C} \qquad \frac{B, \Psi \longrightarrow C}{A \vee B, \Psi \longrightarrow C}$$

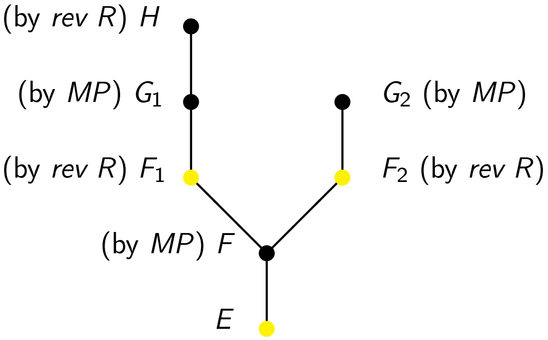
J refutation trees

$RT(E)$ (top-down)



J refutation trees

$RT(E)$ (bottom-up)



Syntactic completeness

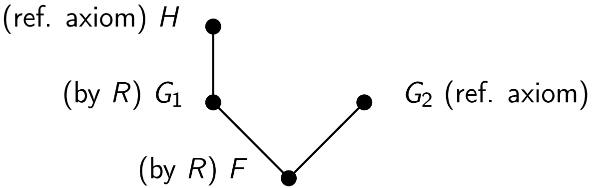
We say that F is *refutable* ($\dashv F$) iff F is derivable from refutation axioms by refutation rules.

Theorem

Let F be a $\mathbf{J}$ *normal form. Then either* $\vdash F$ *or* $\dashv F$*.*

J counter-models

$T(E)$



Remark Every node in $T(E)$ is a special **J** normal form.

J counter-models

$(T(E), V)$

Definition

Let $F = \Sigma \longrightarrow a$ be a node in $T(E)$.
(i) F is abnormal iff F is abnormal (Q).
(ii) $V(A, F) = 1$ iff $A \in \Sigma$ ($A \in VAR$).

J counter-models

Definition

Let F be a node in $RT(F)$. The corresponding point $o(F)$ in $T(F)$ is F if F is a refutation axiom or obtained by R; and it is F' if F is obtained from F' by RMP.

Remark If F is black then $o(F)$ is F; and if F is yellow then $o(F)$ is the first black point above F.

Semantic completeness

Theorem

Let $F = \Sigma \longrightarrow a$ be a node in $RT(E)$. Then Σ is true at $o(F)$ and a is false at $o(F)$.

Corollary

Let A be a formula. $\vdash A$ iff A is valid in all finite $\mathbf{J}$ trees.

History

D. Scott, Completeness proofs for the intuitionistic sentential calculus, 1957.

L. Pinto and R. Dyckhoff, Loop-free construction of counter-models for intuitionistic propositional logic, 1995.

T. Skura, Refutations and proofs in S4, (in) *Proof Theory of Modal Logic*, 1996.

T. Skura, Refutations, proofs, and models in the modal logic K4, SL, 2002.

T. Skura, Intuitionistic Socratic procedures, JANCL, 2005.

T. Skura, *Refutation Methods in Modal Propositional Logic*, 2013.

T. Skura, The greatest paraconsistent analogue of Intuitionistic Logic, PLS, 2017.

Applications

- Constructive completeness proofs that are elegant.
- Refined semantic characterizations by finite tree-type frames.
- Loop-free constructions of counter-models.
- Decision procedures.

Decision procedures

-|

Theorem

Let F be a special **J** *normal form. Then F is non-valid iff either $\{F_1, ..., F_k\}$ is non-valid or G_1 is non-valid or ... or G_k is non-valid.*

So F is non-valid iff some end node is non-valid.

|-

Theorem

Let F be a special **J** *normal form. Then F is valid iff either $\{F_1, G_1\}$ is valid or ... or $\{F_k, G_k\}$ is valid.*

So F is valid iff some end node is valid.

Segerberg's logics

J plus some of the following.

(E)	$p \vee (p \to q)$
(X)	$p \vee \neg p$
(K)	$\neg p \vee \neg\neg p$
(E_1^Q)	$\bot \to p \vee (p \to q)$
(L)	$(p \to q) \vee (q \to p)$
(L')	$\neg p \vee (\bot \to p)$
(L_1^Q)	$\bot \to (p \to q) \vee (q \to p)$
(L^N)	$(p \to q \vee \bot) \vee (q \to p \vee \bot)$

K. Segerberg, Propositional logics related to Heyting's and Johansson's, *Theoria*, 1968.

The logic **JE**

JE is **J** plus
(E) $p \vee (p \to q)$

A **JE** frame is a **J** frame such that the accessibility relation is $=$ (identity).

P. Bernays, H. B. Curry, S. Kanger, S. Kripke, J. Porte, N.C.A. da Costa and J.-Y. Béziau, I. Urbas (see S. Odintsov, *Constructive Negations and Paraconsistency*),
T. Skura, Maximality and refutability, NDJFL, 2004.

JE

F is a **JE** *normal form* iff F is a **J** normal form of rank 0.

Lemma

Every **J** *normal form is* **JE** *equivalent to a conjunction of* **JE** *normal forms.*

$\Sigma \longrightarrow a$ is equivalent to $b \vee (b \to c), \Sigma \longrightarrow a$
(If $\vdash A$ then $B \to C$ is eq. to $A \wedge B \to C$)
which is equivalent to

$(b, \Sigma \longrightarrow a) \wedge ((b \to c), \Sigma \longrightarrow a)$
$(\vdash (A \vee B \to C) \equiv (A \to C) \wedge (B \to C))$

$(b \to c) \to d$ eliminated

$b, (c \to d), \Gamma$ (or $(b \to c), d, \Gamma$) normalized

JE

Corollary

For every formula A there is a conjunction C_A of **JE** *normal forms such that*
$\vdash A$ *iff* $\vdash C_A$.
(In fact, $\vdash A \to C_A$*.)*

JE

Refutation axioms: Every special **JE** normal form F .
(Either F is not valid in T_1 or F is not valid in T_1.)

Refutation rules: None.

Theorem
JE *is characterized by* $\{T_1, T_1\}$.

In other words, $\textbf{JE} = VAL(T_1) \cap VAL T_1)$

The logic **JK**

JK is **J** plus
(K) $\neg p \vee \neg\neg p$

A **JK** frame is a directed **J** frame.

JK normal forms

Let $F = \Sigma \longrightarrow a$ be a **J** normal form.
$\Pi = VAR(F)$.

F is equivalent to $\{\neg b \vee \neg\neg b : b \in \Pi\}, \Sigma \longrightarrow a$.
So F is equivalent to a conjunction of formulas
$\neg\neg\Pi_1 \cup \Delta, \neg\Pi_0 \cup \Gamma \longrightarrow a$
where $\Pi_0 \cap \Pi_1 = \emptyset$ and $\Pi_0 \cup \Pi_1 = \Pi$.

JK normal forms

Definition

A **JK** *normal form* is either a Q **J** normal form or a non-Q **J** normal form $F = \Sigma \longrightarrow a$ that satisfies the following conditions.
(i) $\neg\Pi_0 \subseteq \Sigma$.
(ii) Either
(α) $\neg\neg\Pi_1 \subseteq \Sigma$ and $\bot \to a \in \Sigma$
or
(ω) $\neg\neg\Pi_1 \cap \Sigma = \emptyset$, $\Pi_1 \subseteq \Sigma$, $a = \bot$, and Δ is reduced.
(Here (Π_0, Π_1) is a partition of $VAR(F)$.)

The non-Q **J** normal forms satisfying α will be called α normal forms, and those satisfying ω - ω normal forms.

JK refutation rules

(R_α) $\qquad \dfrac{F_\bot, F_1, ..., F_l}{F}$

where F is a special α normal form, and
$F_\bot$ results from F by replacing a with $\bot$ and $\neg\neg\Pi_1$ with Π_1
(and reducing the antecedent).

(R_ω) $\qquad \dfrac{F_1, ..., F_l}{F}$

where F is a special ω normal form.

JK counter-models

A finite **JK** frame is obtained from a finite **J** tree by picking a normal point ω and extending $\leq$ to $\leq'$ as follows.

$x \leq' y$ iff either $x \leq y$ or (x is normal and $y = \omega$).

Remark Let $A \in VAR$.
If A is true at ω, then $\neg\neg A$ is true at every normal point (because $\neg A$ is false there).

JK counter-models

$RT(E)$

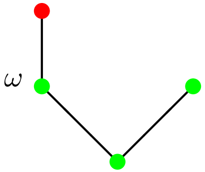
Consider E that is obtained by R_α.

Then, in $T(E)$, $\omega = o(E_\perp)$.

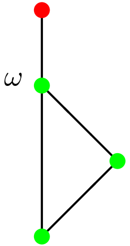
The accessibility relation $\leq$ in $T(E)$ is extended to $\leq'$ as follows.

$\leq' = \leq \cup \{(F, \omega) : F \text{ is normal}\}$

JK counter-models



$T(E)$



$T_K(E)$

Decidability

The question whether the following logics are decidable was left open by Segerberg.

$JL_1^Q, JL_1^Q L', JKL_1^Q, JKL_1^Q L', JL^N L_1^Q, JL^N L_1^Q L', JL_1^Q X;$
$JE_1^Q, JE_1^Q L', JKE_1^Q, JKE_1^Q L', JL^N E_1^Q, JL^N E_1^Q L', JE_1^Q X.$

Theorem

These logics are decidable.

The essentials of efficient decision procedures provided by our refutation systems - see T. Skura, Refutations in Wansing's logic, *Reports on Mathematical Logic* **52** (2017), 83-99.

Nelson's logics

The logic $\mathbf{N4}^{\perp}$
is **Int** plus the axioms for strong negation:

$$\sim\sim p \equiv p$$
$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$
$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$
$$\sim(p \to q) \equiv p \wedge \sim q$$

$\mathbf{N3}$ is $\mathbf{N4}^{\perp}$ plus
$(\mathsf{I}^{\sim})\ p \wedge \sim p \to q$

We also write $\mathcal{N}$ for $\mathbf{N4}^{\perp}$.

S. Odintsov, *Constructive Nagations and Paraconsistency*, 2008.

$\mathcal{N}$ normal forms

$A \Leftrightarrow B = (A \equiv B) \wedge ({\sim} A \equiv {\sim} B)$ (strong equivalence)

(*Replacement*) $(A \Leftrightarrow B) \to (C(A) \equiv C(B))$

We get F_A by using $\Leftrightarrow$ instead of $\equiv$ in Σ_A and reducing $(p_C \otimes p_D) \Leftrightarrow p_B$ (where $B = C \otimes D$ is a subformula of A, and $\otimes \in \{\to, \wedge, \vee\}$) to simpler formulas.

As a result, an $\mathcal{N}$ normal form is just like a **J** normal form but a, b, c, d are (negated) atoms.

We say that an $\mathcal{N}$ normal form F is *inconsistent* iff $A, {\sim} A \in \Gamma$ for some $A \in VAR$.

$\mathcal{N}$ models

$((W, \leq), V)$
$\{0\}$ (false), $\{1\}$ (true), $\emptyset$ (neither), $\{0, 1\}$ (both)

We say that $x \in W$ is *inconsistent* iff some $A \in VAR$ is both true and false at x for some V.