Proofs and refutations getting married

(Some contemplations 24 years later)

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Poznań Reasoning Week – Refutation Symposium 2018
Poznań, September 14, 2018

Outline of the talk

- Introduction: sentential refutation systems.
- Refutation systems for logics with finite tree-model property.
- Refutation systems for logics and theories of finite models.
- Combining refutations and derivations. Hybrid deduction systems.
- Towards a meta-proof theory of hybrid deduction systems.
- Concluding remarks.

Introduction: semantic and deductive refutability

Consider a logical system L with semantics defining validity $\models$,
and a complete deductive system D for L, with derivation relation $\vdash_D$.

Then, for any L-formula A, derivability $\vdash_D A$ corresponds to validity $\models A$.

The formula A is semantically refutable in L iff it is not valid, i.e. $\not\models A$.

The matching syntactic notion is ***not*** non-derivability of A, i.e. $\nvdash_D A$.

It is rather the notion of "deductive refutability".

A new symbol: $\dashv$, is needed for it, such that

$$\dashv A$$

means "A is provably refutable", i.e. "the non-validity of A is deduced".

Hereafter we will also read it as "the refutation of A is derived".

Thus, the notion of "deductive refutation" arises.

It is usually relativised to the deductive system D, which it involves.

Introduction:
Sentential refutation systems and basic concepts

(Hybrid) rules of refutation inference:

$$\frac{\vdash A_1, \ldots \vdash A_m, \ \dashv B_1, \ldots \dashv B_n}{\dashv C}$$

Sentential refutation system (based on a deductive system **S**):
a system of refutation inference rules.

- Correctness (of $\vdash$), and Ł-correctness (of $\dashv$).
- Completeness (of $\vdash$), and refutation-completeness (of $\dashv$).
- Ł-completeness and adequacy (all of the above).

Introduction: sentential refutation system for classical propositional logic

A refutation system for the Classical Propositional Calculus CPC($\dashv$) (due to Łukasiewicz):

Axiom: $\dashv \bot$.

Rules:

Reverse Substitution **RS**:

$$\frac{\dashv \sigma(\phi)}{\dashv \phi}$$

for any uniform substitution σ.

Modus Tollens **MT**:

$$\frac{\vdash \phi \to \psi, \ \dashv \psi}{\dashv \phi}.$$

NB: a refutation system can be Ł-complete for more than one logic.

Example: CPC($\dashv$) is Ł-complete not only for the CPC, but also for both maximal normal modal logics $\mathbf{K} + \Box\bot$ and $\mathbf{K} + (p \leftrightarrow \Box p)$.

Refutation systems for logics with the FMP

Ł-complete refutation systems for some modal logics

VG'1994, "*Refutation Systems in Modal Logic*", Studia Logica, vol. 53.

Ł-complete refutation system for **K**:

$$\mathsf{CPC}(\dashv) + \dashv \Diamond\top +$$

$$\mathbf{R_K}: \quad \frac{\dashv \alpha, \ \dashv \psi \vee \theta_1, \ldots, \dashv \psi \vee \theta_k}{\dashv \alpha \vee \Diamond\psi \vee (\Box\theta_1 \vee \ldots \vee \Box\theta_k)}$$

Proof: **K** is complete for the class of all finite irreflexive and intransitive trees.

Ł-complete refutation system for **GL** (=**KW**) $= \mathbf{K} + \Box(\Box p \to p) \to \Box p$:

$$\mathsf{CPC}(\dashv) + \dashv \Diamond\top +$$

$$\mathbf{R_{GL}}: \quad \frac{\dashv \alpha, \ \dashv \psi \vee \Diamond\psi \vee \theta_1, \ldots, \dashv \psi \vee \Diamond\psi \vee \theta_k}{\dashv \alpha \vee \Diamond\psi \vee (\Box\theta_1 \vee \ldots \vee \Box\theta_k)}$$

Proof: **GL** is complete for the class of all finite irreflexive and transitive trees.

For more such results (incl. K4, S4, S4Grz, etc) see the paper.

Refutation systems for modal logics with finite tree-model property

The key for these results is a finite tree-model property: every satisfiable formula of the logic is satisfiable in a tree-like model for that logic.

A more complex formula can be satisfied in a tree by satisfying its simpler 'components' in suitable subtrees.

That generates respective refutation inference rules.

Ł-completeness follows from the finite tree-model property.

References: Skura, 1994, 2004, 2013.

Generic refutation systems for modal logics of finite frames: preparation

Consider a finite Kripke frame $\mathcal{F} = \langle W, R \rangle$, where $W = \{w_1, \ldots, w_k\}$.

Let $P = \{p_1, \ldots, p_k\}$ be a set of fixed distinct propositional variables.

Let $\mathcal{F}^P = \langle \mathcal{F}, V^P \rangle$ be the Kripke model where $V^P(p_i) = \{w_i\}$, for $i = 1, \ldots, k$, and $V^P(q) = \emptyset$ for any other propositional variable.

Further, for any $n \in \mathbb{N}$ and $w_i \in W$, let $\chi_n(\mathcal{F}^P, w_i)$ be the characteristic formula of depth n of the pointed Kripke model $(\mathcal{F}^P, w_i)$.

Thus, for any pointed Kripke model (M, w), it holds that $M, w \models \chi_n(\mathcal{F}^P, w)$ iff (M, w) and $(\mathcal{F}^P, w)$ are n-bisimilar, hence, satisfying the same modal formulae of modal depth up to n.

Lastly, let $\Sigma_\vee(P)$ be the set of all substitutions that replace any propositional variable by a disjunction of variables from P.

Generic refutation systems for modal logics of finite frames

Theorem

CPC($\dashv$) *extended with the refutation rule schema*

$$\textbf{\textit{Ref}}_{\mathcal{F}} \qquad \frac{\vdash_{\mathbf{K}} \chi_n(\mathcal{F}^P, w) \to \neg\sigma(\phi)}{\dashv \phi}$$

where $w \in W$, ϕ is a modal formula of depth n, and $\sigma \in \Sigma_{\vee}(P)$, is Ł-complete for the normal modal logic $\mathbf{K}_{\mathcal{F}}$ of all validities in $\mathcal{F}$.

Proof sketch: ϕ is falsifiable at some world w for some valuation in $\mathcal{F}$ iff a suitable substitution $\sigma \in \Sigma_{\vee}(P)$ fails at w in $\mathcal{F}^P$, and therefore fails at every pointed model which is n-bisimilar to $(\mathcal{F}^P, w)$.

Shortcomings:

- Requires a complete deductive system in the background.
- Infinitely many and complex refutation rules, with no subformula property.
- These rules do not build the refuted formulae step-by step, but produce them at once.

An alternative construction of a refutation system for the modal logic of a finite frame

Again, consider a finite Kripke frame $\mathcal{F} = \langle W, R \rangle$, where $W = \{w_1, \dots, w_k\}$.

Idea sketch: add a 'local' refutation rule for every state of $\mathcal{F}$, that generate the formulae satisfiable at that state from the formulae satisfiable at its successors in the frame.

Details to follow in a paper in preparation.

Generalising: refutation systems for modal logics with finite model property

The result extends to the modal logic $\mathbf{K}_{\mathcal{T}}$ of any r.e. set $\mathcal{T}$ of finite Kripke frames, by adding refutation rules $\mathbf{Ref}_{\mathcal{F}}$ for each frame $\mathcal{F} \in \mathcal{T}$.

In particular, it extends to any finitely axiomatized normal modal logic with finite model property.

Likewise, to refutation systems for FO theories with finite model property.

Refutation system for the FO theory of a finite model

Consider a finite FO model $\mathcal{F}$ and let $\Delta_{\mathcal{F}}$ be a FO sentence that describes $\mathcal{F}$ up to isomorphism.

Let FO($\vdash$) be any complete deductive system for FOL.

Theorem

FO($\vdash$) + CPC($\dashv$), *extended with the refutation rule schema*

$$\textbf{Ref}_{\mathcal{F}} \qquad \frac{\vdash \Delta_{\mathcal{F}} \to \neg\phi}{\dashv \phi}$$

and the refutation rules ***Reverse generalisation****:*

$$\textbf{RG} \quad \frac{\dashv \forall x \phi}{\dashv \phi}$$

and ***Reverse instantiation****:*

$$\textbf{RI} \quad \frac{\dashv \phi(c)}{\dashv \phi(x)}$$

is Ł-complete for the FO theory TH($\mathcal{F}$) of all validities in $\mathcal{F}$.

Refutation systems for FO in the finite

Validity in the finite in FOL is not r.e. but is co-r.e.

Therefore, refutability in $\mathsf{FOL_{fin}}$ is r.e.

(T. Skura, 2018): a refutation system for the validity in the finite in FOL: $\mathsf{FOL_{fin}}(\dashv)$, extending $\mathsf{CPC}(\dashv)$ with **RG** and **RI**, plus the rules $\mathbf{R}_n$:

$$\frac{\dashv \phi}{\dashv \delta_n \to \phi}$$

for each $n \in \mathbb{N}$, where $\delta_n = \forall x_{n+1}(x_{n+1} = x_1 \vee \ldots \vee x_{n+1} = x_n)$ and ϕ is a quantifier-free formula containing n terms. (No function symbols.)

Other, earlier proposals:

- T. Hailperin: "A Complete Set of Axioms for Logical Formulas Invalid in Some Finite Domains", MLQ, 1961.
  FO language with no equality, constant or functional symbols.
  All open non-tautologies taken as axioms.
  Three rules of inference, for the quantifiers.
- M. Tiomkin, LICS 1988: a calculus of refutable sequents, claimed complete for refutability in the finite.

Refutation systems for other non-axiomatized logics

1. The modal logic $\mathsf{SUB}_{\mathbb{N}}$ of subintervals relation on $\mathbb{N}$.

Marcinkowski and Michaliszyn, LICS'2011: $\mathsf{SUB}_{\mathbb{N}}$ is undecidable.

But, it has the FMP, by definition. (Hence, it not r.e.)

2. Medvedev's modal logic **M**:
the logic of all finite frames of the type $\langle \mathcal{P}^{+}(X), \supseteq \rangle$.

Still unknown whether it is decidable (and hence, whether it is r.e.).

No complete refutation systems for any of these have been designed yet(?)

Hybrid deduction rules

Combining refutations and derivations: hybrid rules

Hybrid rules for deduction and refutation inference:

$$\frac{\vdash \phi_1, \ldots \vdash \phi_m,\ \dashv \psi_1, \ldots \dashv \psi_n}{\vdash \theta} \qquad \frac{\vdash \phi_1, \ldots \vdash \phi_m,\ \dashv \psi_1, \ldots \dashv \psi_n}{\dashv \theta}$$

Example: **Modus Tolens**:

$$\frac{\vdash \phi \to \psi,\ \dashv \psi}{\dashv \phi}$$

A general idea: use suitable meta-properties of the given logic to justify hybrid inference rules.

Example: using/stating consistency:

$$(\textbf{Cons}) \qquad \frac{\vdash \neg\phi}{\dashv \phi}$$

Example: using the Disjunction property of INT (and other logics, incl. **M**):

$$\frac{\vdash \phi \vee \psi,\ \dashv \psi}{\vdash \phi}$$

Rules combining refutations and derivations more examples

Disjunction rule (for logics with the Disjunction property):

$$\frac{\dashv \phi, \ \dashv \psi}{\dashv \phi \vee \psi}$$

Box refutation rule:

$$\frac{\dashv \phi}{\dashv \Box\phi}$$

Sound e.g. in **K** and other logics complete for classes of frames closed under extending with roots.

Modal disjunction rules:

$$\frac{\vdash \Box\phi \vee \Box\psi, \ \dashv \psi}{\vdash \phi}$$

Sound e.g. in **K** and other logics with tree-model property.

Designing hybrid natural deduction rules

Every pure inference rule of Natural Deduction (ND) can produce one or more hybrid rules, by swapping a premise with the conclusion and replacing $\vdash$ with $\dashv$ in both.

Introduction rules for $\vdash$ become elimination rules for $\dashv$ and vice versa.
NB: the open assumptions need to be listed in the rules.

Examples: transforming the introduction rules of ND for Classical logic:

$$\frac{\Gamma \vdash \phi,\ \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi} \ \Rightarrow \ \frac{\Gamma \dashv \phi \wedge \psi,\ \Gamma \vdash \psi}{\Gamma \dashv \phi}, \ \frac{\Gamma \vdash \phi,\ \Gamma \dashv \phi \wedge \psi}{\Gamma \dashv \psi}$$

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi} \ \Rightarrow \ \frac{\Gamma \dashv \phi \vee \psi}{\Gamma \dashv \phi}; \quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi} \ \Rightarrow \ \frac{\Gamma \dashv \phi \vee \psi}{\Gamma \dashv \psi}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi} \ \Rightarrow \ \frac{\Gamma \dashv \phi \to \psi}{\Gamma, \phi \dashv \psi}; \quad \frac{\Gamma, \phi \vdash \bot}{\Gamma \vdash \neg\phi} \ \Rightarrow \ \frac{\Gamma \dashv \neg\phi}{\Gamma, \phi \dashv \bot}$$

$$\frac{\Gamma \vdash \phi[c/x]}{\Gamma \vdash \forall x \phi(x)} \ \Rightarrow \ \frac{\Gamma \dashv \forall x \phi(x)}{\Gamma \dashv \phi[c/x]}; \quad \frac{\Gamma \vdash \phi[t/x]}{\Gamma \vdash \exists x \phi(x)} \ \Rightarrow \ \frac{\Gamma \dashv \exists x \phi(x)}{\Gamma \dashv \phi[t/x]}$$

where c is a constant symbol, not occurring in $\phi(x)$, nor in any open assumption used in the derivation of $\phi[c/x]$; t is any term t free for x in $\phi(x)$.

Transforming the elimination rules of ND for Classical logic

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \wedge \psi} \Rightarrow \frac{\Gamma \dashv \phi \wedge \psi}{\Gamma \dashv \phi}; \quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi} \Rightarrow \frac{\Gamma \dashv \phi \wedge \psi}{\Gamma \dashv \psi}$$

$$\frac{\Gamma, \phi \vdash \theta,\ \Gamma, \psi \vdash \theta}{\Gamma, \phi \vee \psi \vdash \theta} \Rightarrow \frac{\Gamma, \phi \vee \psi \dashv \theta,\ \Gamma, \psi \vdash \theta}{\Gamma, \phi \dashv \theta}, \quad \frac{\Gamma, \phi \vdash \theta,\ \Gamma, \phi \vee \psi \dashv \theta}{\Gamma, \psi \dashv \theta}$$

$$\frac{\Gamma \vdash \phi,\ \Gamma \vdash \phi \to \psi}{\Gamma \vdash \psi} \Rightarrow \frac{\Gamma \dashv \psi,\ \Gamma \vdash \phi \to \psi}{\Gamma \dashv \phi}, \quad \frac{\Gamma \vdash \phi, \Gamma \dashv \psi}{\Gamma \dashv \phi \to \psi}$$

$$\frac{\Gamma \vdash \phi,\ \Gamma \vdash \neg\phi}{\Gamma \vdash \bot} \Rightarrow \frac{\Gamma \dashv \bot,\ \Gamma \vdash \neg\phi}{\Gamma \dashv \phi}, \quad \frac{\Gamma \vdash \phi,\ \Gamma \dashv \bot}{\Gamma \dashv \neg\phi}$$

$$\frac{\Gamma \vdash \forall x \phi(x)}{\Gamma \vdash \phi[t/x]} \Rightarrow \frac{\Gamma \dashv \phi[t/x]}{\Gamma \dashv \forall x \phi(x)}, \quad \frac{\Gamma, \phi[c/x] \vdash \psi}{\Gamma, \exists x \phi(x) \vdash \psi} \Rightarrow \frac{\Gamma, \exists x \phi(x) \dashv \psi}{\Gamma, \phi[c/x] \dashv \psi},$$

where t is any term t free for x in $\phi(x)$; c is a constant symbol, not occurring in $\phi(x)$, nor in ψ or in any open assumption in the derivation of ψ, except for $\phi[c/x]$.

Transforming "Ex falso quodlibet":

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \phi} \quad \Rightarrow \quad \frac{\Gamma \dashv \phi}{\Gamma \dashv \bot}$$

Transforming "Reductio ad absurdum":

$$\frac{\Gamma, \neg\phi \vdash \bot}{\Gamma \vdash \phi} \quad \Rightarrow \quad \frac{\Gamma \dashv \phi}{\Gamma, \neg\phi \dashv \bot}$$

Hybrid deduction systems

Hybrid deduction systems

Hybrid deduction system: for combined derivations and refutations inferences, involving a set of hybrid rules.

The explicit idea goes back to Łukasiewicz and (for me) to VG' 1994.

Important class: symmetric inference systems, Skura' 2004.
Involve only pure derivation rules plus axioms and rejected formulae (aka, "anti-axioms").

Other related proposals:

- Caferra and Peltier, 2008: "Accepting/Rejecting Propositions From Accepted/Rejected Propositions: A Unifying Overview"
- Citkin, 2015: "A Meta-Logic Of Inference Rules: Syntax"

Designing hybrid deduction systems from natural deduction systems

Every system of Natural Deduction **ND** can be uniformly extended to a system of Hybrid Natural Deduction **H(ND)**, by adding all transformed hybrid rules.

The question of Ł-completeness arises.

The proof of Ł-completeness of **H(ND)** amounts to formalising the proof of soundness and completeness of **ND** in **H(ND)**,
by replacing throughout $\nvdash$ with $\dashv$.

This should be doable for decidable logics, but not in general.

Conjecture: The system **H(ND)** for FOL is refutation-complete for the set of FO sentences that are refutable in the finite.

Towards a meta-proof theory of hybrid deduction systems

Add a new meta-symbol **F**, for "absurd", or "falsum" in the meta-language.

Now, new hybrid rules can be added:

Cons, stating consistency:

$$\frac{\vdash \phi, \ \dashv \phi}{\mathbf{F}}$$

EFQ: "Ex falso Ł-quodlibet":

$$\frac{\mathbf{F}}{\vdash \phi}, \qquad \frac{\mathbf{F}}{\dashv \phi}$$

Ł-Comp: "Ł-completeness" and **Ł-RAA**: "Ł-Reductio ad absurdum"

$$(\textbf{Ł-Comp}) \quad \frac{\begin{matrix}[\vdash \phi] \\ \vdots \\ \mathbf{F}\end{matrix}}{\dashv \phi} \qquad (\textbf{Ł-RAA}) \quad \frac{\begin{matrix}[\dashv \phi] \\ \vdots \\ \mathbf{F}\end{matrix}}{\vdash \phi}$$

Do these strengthen the deductive power of the hybrid deduction system?

Towards a meta-proof theory of hybrid deduction systems, continued

Deductive completeness + Ł-completeness can now be stated as additional hybrid rules:

$$(\textbf{Ded})\quad \dfrac{\begin{matrix}[\dashv \phi] & [\vdash \phi] \\ \vdots & \vdots \\ \vdash \psi & \vdash \psi\end{matrix}}{\vdash \psi} \qquad\qquad (\textbf{Ref})\quad \dfrac{\begin{matrix}[\dashv \phi] & [\vdash \phi] \\ \vdots & \vdots \\ \dashv \psi & \dashv \psi\end{matrix}}{\dashv \psi}$$

Do these strengthen the deductive power of the hybrid deduction system?

Do they bring about deductive completeness or Ł-completeness, when it does not hold without them?

Concluding remarks

Follow-ups: a lot!

- Prove Ł-completeness of **H(ND)** for classical propositional logic.
- Prove refutation-completeness of the system **H(ND)** for FOL for the non-validities in the finite.
- Design Ł-complete hybrid deductive systems for FO modal logics.
- Develop the meta-proof theory of hybrid deduction systems.
- Relate hybrid deduction systems with tableaux.
- Applications?

The end